[Truncated because of teaching evaluations...] *Recap:*

Recall the product formula for $\mathbb{Q}$:

$$\prod_{v \le \infty} |x|_v = 1 \qquad \text{(for } x \in \mathbb{Q}^\times\text{)}$$

That is, with $|*|_\infty$ the 'usual' absolute value on $\mathbb{R}$,

$$|x|_\infty \cdot \prod_{p \text{ prime}} |x|_p = 1 \qquad \text{(for } x \in \mathbb{Q}^\times\text{)}$$

To have the product formula hold for *number fields* k: for $\mathfrak{p}$ lying over p, letting $k_\mathfrak{p}$ be the $\mathfrak{p}$-adic completion of k and Q_p the usual p-adic completion of $\mathbb{Q}$,

$$|x|_\mathfrak{p} = |N^{k_\mathfrak{p}}_{\mathbb{Q}_p} x|_p = N\mathfrak{p}^{-\mathrm{ord}_\mathfrak{p} x}$$

Similarly, for *archimedean* k_v, define (or renormalize)

$$|x|_v = |N^{k_v}_{\mathbb{R}} x|_\infty$$

This product-formula normalization of the norm on $\mathbb{C}$ (harmlessly) fails to satisfy the triangle inequality:

$$|x|_{\mathbb{C}} \;=\; |N^{\mathbb{C}}_{\mathbb{R}} x|_\infty \;=\; x \cdot \overline{x} \;=\; \textit{square} \text{ of usual complex abs value}$$

For example,

$$|2|_{\mathbb{C}} \;=\; |N^{\mathbb{C}}_{\mathbb{R}} 2|_{\mathbb{R}} \;=\; |4|_{\mathbb{R}} \;=\; 4 \;>\; 1+1 \;=\; |1|_{\mathbb{C}} + |1|_{\mathbb{C}}$$

For **function fields** $k = \mathbb{F}_q(x)$, for p-adic v associated to non-zero prime $\mathfrak{p} = \varpi \mathbb{F}_q[x]$, the same sort of definition of norm is appropriate:

$$|f|_v \;=\; N\mathfrak{p}^{-\mathrm{ord}_{\mathfrak{p}} f} \;=\; q^{-\deg \varpi \cdot \mathrm{ord}_{\mathfrak{p}} f}$$

The *infinite* norm $|*|_\infty$ corresponding to the prime ideal $\mathfrak{q}$ generated by $1/x$ in $\mathfrak{o}_\infty = \mathbb{F}_q[1/x]$, is

$$|f|_v \;=\; q^{+\deg f} \;=\; |\mathfrak{o}_\infty/\mathfrak{q}|^{-\mathrm{ord}_{\mathfrak{q}} f}$$

since $a_n x^n + \ldots + a_o = (\frac{1}{x})^{-n}(a_n + \ldots + a_o(\frac{1}{x})^n)$

Theorem: *(Product formula for number fields)*

$$\prod_{\text{places } w \text{ of } k} |x|_w \;=\; \prod_{\text{places } v \text{ of } \mathbb{Q}} \;\prod_{w|v} |N^{k_w}_{\mathbb{Q}_v}(x)|_v \;=\; 1 \qquad (\text{for } x \in k^\times)$$

because, for K/k an extension of number fields, the *global* norm is the product of the *local* norms:

$$\prod_{w|v} N^{K_w}_{k_v}(x) \;=\; N^K_k(x) \qquad (\text{for } x \in K, \text{ abs value } v \text{ of } k)$$

Corollaries of proof: The sum of the *local* degrees is the *global* degree:

$$\sum_{w|v} [K_w : k_v] = [K : k]$$

The global trace is the sum of the local traces:

$$\mathrm{tr}^K_k(x) \;=\; \mathrm{tr}^{K_w}_{k_v}(x) \qquad (\text{for } x \in K)$$

Why do we care about formulas $\prod_v$ *symbol*$_v(x) = 1$?

The **idele group** $\mathbb{J} = \mathbb{J}_k$ of k is a colimit over finite sets S of places containing archimedean places:

$$\mathbb{J} \;=\; \mathbb{J}_k \;=\; \mathrm{colim}_S\Big(\prod_{v\in S} k_v^\times \times \prod_{v\notin S} \mathfrak{o}_v^\times\Big)$$

The idele group *surjects* to the group of fractional ideals of k, by

$$\alpha = \{\alpha_v\} \;\longrightarrow\; \prod_{v<\infty} \Big((\alpha_v \cdot \mathfrak{o}_v) \cap k\Big)$$

$k^\times$ maps to *principal* fractional ideals, so the **idele class group** $\mathbb{J}/k^\times$ surjects to the *ideal class group* C_k. It also parametrizes *generalized* class groups.

An **idele class character**, or **Hecke character**, or **grossencharacter**, is a continuous group hom $\mathbb{J}/k^\times \to \mathbb{C}^\times$. *Some* of these characters arise from composition with *ideal class group* characters χ, by

$$\mathbb{J}/k^\times \longrightarrow C_k \xrightarrow{\;\chi\;} \mathbb{C}^\times$$

The *product formula* asserts that the **idele norm**

$$x = \{x_v\} \longrightarrow |x| = \prod_{v \le \infty} |x_v|_v \qquad \text{(for } x \in \mathbb{J}_k\text{)}$$

factors through $\mathbb{J}/k^\times$. Thus, for $s \in \mathbb{C}$, we have an idele class character

$$x \longrightarrow |x|^s \qquad \text{(for } x \in \mathbb{J}/k^\times\text{)}$$

These characters enter the Iwasawa-Tate modern version of Riemann's argument for meromorphic continuation and functional equation of zeta functions and (abelian) L-functions.

Proving that an infinite product of almost-all 1's is equal to 1 should remind us of *reciprocity laws*, although reciprocity laws are subtler than the product formula. Recall

$$\begin{gathered} \text{quadratic norm residue symbols} \subset \text{idele class characters} \\ \Downarrow \\ \text{quadratic Hilbert symbol reciprocity} \\ \Downarrow \\ \text{quadratic reciprocity (general)} \end{gathered}$$

Classification of completions *(often attributed to Ostrowski) :* The topologically inequivalent (non-discrete) norms on $\mathbb{Q}$ are the usual $\mathbb{R}$ norm and the p-adic $\mathbb{Q}_p$'s.

Proof: Let $|*|$ be a norm on $\mathbb{Q}$. It turns out (intelligibly, if we guess the answer) that the watershed is whether $|*|$ is *bounded* or *unbounded* on $\mathbb{Z}$. That is, the statement of the theorem could be sharpened to say: norms on $\mathbb{Q}$ bounded on $\mathbb{Z}$ are topologically equivalent to p-adic norms, and norms unbounded on $\mathbb{Z}$ are topologically equivalent to the norm from $\mathbb{R}$.

For $|*|$ bounded on $\mathbb{Z}$, in fact $|x| \leq 1$ for $x \in \mathbb{Z}$, since otherwise $|x^n| = |x|^n \to +\infty$ as $n \to +\infty$.

To say that $|*|$ is *bounded* on $\mathbb{Z}$, but *not discrete*, implies $|x| < 1$ for some $x \in \mathbb{Z}$, since otherwise $d(x,y) = |x - y| = 1$ for $x \neq y$, giving the discrete topology.

Then, by unique factorization, $|p| < 1$ for some prime number p. If there were a second prime q with $|q| < 1$, with $a, b \in \mathbb{Z}$ such that $ap^m + bq^n = 1$ for positive integers m, n, then

$$1 \;=\; |1| \;=\; |ap^m + bq^n| \;\le\; |a| \cdot |p|^m + |b| \cdot |q|^n \;\le\; |p|^m + |q|^n$$

This is impossible if *both* $|p| < 1$ and $|q| < 1$, by taking m, n large. Thus, for $|*|$ bounded on $\mathbb{Z}$, there is a unique prime p such that $|p| < 1$. Up to normalization, such a norm is the p-adic norm.

Next, claim that if $|a| \le 1$ for some $1 < a \in \mathbb{Z}$, then $|*|$ is *bounded* on $\mathbb{Z}$. Given $1 < b \in \mathbb{Z}$, write b^n in an a-ary expansion

$$b^n \;=\; c_o + c_1 a + c_2 a^2 + \ldots + c_\ell a^\ell \qquad\qquad (\text{with } 0 \le c_i < a)$$

and apply the triangle inequality,

$$|b|^n \;\le\; (\ell + 1) \cdot \underbrace{(1 + \ldots + 1)}_{a} \;\le\; (n \log_a b + 1) \cdot a$$

Taking n^{th} roots and letting $n \to +\infty$ gives $|b| \le 1$, and $|*|$ is bounded on $\mathbb{Z}$.

The remaining scenario is $|a| \ge 1$ for $a \in \mathbb{Z}$.... [cont'd]