Recap:
Recall the product formula for $\mathbb{Q}$:
\[
\prod_{v \leq \infty} |x|_v = 1 \quad \text{(for } x \in \mathbb{Q}^\times \text{)}
\]
That is, with $|\ast|_\infty$ the ‘usual’ absolute value on $\mathbb{R}$,
\[
|x|_\infty \cdot \prod_{p \text{ prime}} |x|_p = 1 \quad \text{(for } x \in \mathbb{Q}^\times \text{)}
\]
To have the product formula hold for number fields $k$: for $\mathfrak{p}$ lying over $p$, letting $k_\mathfrak{p}$ be the $\mathfrak{p}$-adic completion of $k$ and $\mathbb{Q}_p$ the usual $p$-adic completion of $\mathbb{Q}$,
\[
|x|_\mathfrak{p} = |N_{\mathbb{Q}_p}^k x|_p = N_{\mathbb{Q}_p}^{-\text{ord}_\mathfrak{p} x}
\]
Similarly, for archimedean $k_v$, define (or renormalize)
\[
|x|_v = |N_{\mathbb{R}}^k x|_\infty
\]
This product-formula normalization of the norm on \( \mathbb{C} \) (harmlessly) fails to satisfy the triangle inequality:

\[
|x|_\mathbb{C} = |N^\mathbb{C}_\mathbb{R} x|_\infty = x \cdot \overline{x} = \text{square of usual complex abs value}
\]

For example,

\[
|2|_\mathbb{C} = |N^\mathbb{C}_\mathbb{R} 2|_\mathbb{R} = |4|_\mathbb{R} = 4 > 1 + 1 = |1|_\mathbb{C} + |1|_\mathbb{C}
\]

For function fields \( k = \mathbb{F}_q(x) \), for \( p \)-adic \( v \) associated to non-zero prime \( p = \varpi \mathbb{F}_q[x] \), the same sort of definition of norm is appropriate:

\[
|f|_v = N p^{-\text{ord}_p f} = q^{-\deg \varpi \cdot \text{ord}_p f}
\]

The infinite norm \( |*|_\infty \) corresponding to the prime ideal \( q \) generated by \( 1/x \) in \( \mathfrak{o}_\infty = \mathbb{F}_q[1/x] \), is

\[
|f|_v = q^{\deg f} = |\mathfrak{o}_\infty/q|^{-\text{ord}_q f}
\]

since \( a_n x^n + \ldots + a_o = (\frac{1}{x})^{-n}(a_n + \ldots + a_o(\frac{1}{x})^n) \)
**Theorem:** (Product formula for number fields)

\[
\prod_{\text{places } w \text{ of } k} |x|_w = \prod_{\text{places } v \text{ of } \mathbb{Q}} \prod_{w|v} |N_{\mathbb{Q},v}^K(x)|_v = 1 \quad \text{(for } x \in k^\times)\]

because, for $K/k$ an extension of number fields, the *global* norm is the product of the *local* norms:

\[
\prod_{w|v} N_{k,v}^K(x) = N_k^K(x) \quad \text{(for } x \in K, \text{ abs value } v \text{ of } k)\]

**Corollaries of proof:** The sum of the *local* degrees is the *global* degree:

\[
\sum_{w|v} [K_w : k_v] = [K : k]\]

The global trace is the sum of the local traces:

\[
\text{tr}_K^K(x) = \text{tr}_K^K(x) \quad \text{(for } x \in K)\]
Why do we care about formulas $\prod_v \text{symbol}_v(x) = 1$?

The **idele group** $\mathbb{J} = \mathbb{J}_k$ of $k$ is a colimit over finite sets $S$ of places containing archimedean places:

$$\mathbb{J} = \mathbb{J}_k = \text{colim}_S \left( \prod_{v \in S} k_v^\times \times \prod_{v \not\in S} \mathfrak{o}_v^\times \right)$$

The idele group *surjects* to the group of fractional ideals of $k$, by

$$\alpha = \{\alpha_v\} \rightarrow \prod_{v < \infty} \left( (\alpha_v \cdot \mathfrak{o}_v) \cap k \right)$$

$k^\times$ maps to *principal* fractional ideals, so the **idele class group** $\mathbb{J}/k^\times$ surjects to the **ideal class group** $C_k$. It also parametrizes *generalized* class groups.

An **idele class character**, or **Hecke character**, or **grosencharacter**, is a continuous group hom $\mathbb{J}/k^\times \rightarrow \mathbb{C}^\times$. Some of these characters arise from composition with *ideal class group* characters $\chi$, by

$$\mathbb{J}/k^\times \xrightarrow{} C_k \xrightarrow{\chi} \mathbb{C}^\times$$
The **product formula** asserts that the **idele norm**

\[ x = \{x_v\} \longrightarrow |x| = \prod_{v \leq \infty} |x_v|_v \quad \text{(for } x \in \mathbb{J}_k) \]

factors through \( \mathbb{J}/k^\times \). Thus, for \( s \in \mathbb{C} \), we have an idele class character

\[ x \longrightarrow |x|^s \quad \text{(for } x \in \mathbb{J}/k^\times) \]

These characters enter the Iwasawa-Tate modern version of Riemann’s argument for meromorphic continuation and functional equation of zeta functions and (abelian) \( L \)-functions.

Proving that an infinite product of almost-all 1’s is equal to 1 should remind us of **reciprocity laws**, although reciprocity laws are subtler than the product formula. Recall

\[
\begin{align*}
\text{quadratic norm\ residue\ symbols} & \subset \text{idele\ class\ characters} \\
\downarrow
\text{quadratic\ Hilbert\ symbol\ reciprocity} & \downarrow
\text{quadratic\ reciprocity\ (general)}
\end{align*}
\]
Classification of completions (often attributed to Ostrowski) :
The topologically inequivalent (non-discrete) norms on \( \mathbb{Q} \) are the usual \( \mathbb{R} \) norm and the \( p \)-adic \( \mathbb{Q}_p \)'s.

Proof: Let \(|*|\) be a norm on \( \mathbb{Q} \). It turns out (intelligibly, if we guess the answer) that the watershed is whether \(|*|\) is bounded or unbounded on \( \mathbb{Z} \). That is, the statement of the theorem could be sharpened to say: norms on \( \mathbb{Q} \) bounded on \( \mathbb{Z} \) are topologically equivalent to \( p \)-adic norms, and norms unbounded on \( \mathbb{Z} \) are topologically equivalent to the norm from \( \mathbb{R} \).

For \(|*|\) bounded on \( \mathbb{Z} \), in fact \(|x| \leq 1\) for \( x \in \mathbb{Z} \), since otherwise \(|x^n| = |x|^n \rightarrow +\infty\) as \( n \rightarrow +\infty\).

To say that \(|*|\) is bounded on \( \mathbb{Z} \), but not discrete, implies \(|x| < 1\) for some \( x \in \mathbb{Z} \), since otherwise \( d(x,y) = |x-y| = 1\) for \( x \neq y \), giving the discrete topology.
Then, by unique factorization, $|p| < 1$ for some prime number $p$. If there were a second prime $q$ with $|q| < 1$, with $a, b \in \mathbb{Z}$ such that $ap^m + bq^n = 1$ for positive integers $m, n$, then

$$1 = |1| = |ap^m + bq^n| \leq |a| \cdot |p|^m + |b| \cdot |q|^n \leq |p|^m + |q|^n$$

This is impossible if both $|p| < 1$ and $|q| < 1$, by taking $m, n$ large. Thus, for $|\ast|$ bounded on $\mathbb{Z}$, there is a unique prime $p$ such that $|p| < 1$. Up to normalization, such a norm is the $p$-adic norm.

Next, claim that if $|a| \leq 1$ for some $1 < a \in \mathbb{Z}$, then $|\ast|$ is bounded on $\mathbb{Z}$. Given $1 < b \in \mathbb{Z}$, write $b^n$ in an $a$-ary expansion

$$b^n = c_o + c_1a + c_2a^2 + \ldots + c_\ell a^\ell \quad \text{(with } 0 \leq c_i < a)$$

and apply the triangle inequality,

$$|b|^n \leq (\ell + 1) \cdot (1 + \ldots + 1) \leq (n \log_a b + 1) \cdot a$$

Taking $n^{th}$ roots and letting $n \to +\infty$ gives $|b| \leq 1$, and $|\ast|$ is bounded on $\mathbb{Z}$.

The remaining scenario is $|a| \geq 1$ for $a \in \mathbb{Z}$. [cont’d]