Product formula, approximation, ... *[cont'd]*

For **function fields** $k = \mathbb{F}_q(x)$, for p-adic v associated to non-zero prime $\mathfrak{p} = \varpi\mathbb{F}_q[x]$, the same sort of definition of norm is appropriate:

$$|f|_v \;=\; N\mathfrak{p}^{-\mathrm{ord}_\mathfrak{p} f} \;=\; q^{-\deg \varpi \cdot \mathrm{ord}_\mathfrak{p} f}$$

The *infinite* norm $|*|_\infty$ corresponding to the prime ideal $\mathfrak{q}$ generated by $1/x$ in $\mathfrak{o}_\infty = \mathbb{F}_q[1/x]$, is

$$|f|_v \;=\; q^{+\deg f} \;=\; |\mathfrak{o}_\infty/\mathfrak{q}|^{-\mathrm{ord}_\mathfrak{q} f}$$

since $a_n x^n + \ldots + a_o = (\frac{1}{x})^{-n}(a_n + \ldots + a_o(\frac{1}{x})^n)$

Theorem: *(Product formula for number fields)*

$$\prod_{\text{places } w \text{ of } k} |x|_w \;=\; \prod_{\text{places } v \text{ of } \mathbb{Q}} \;\prod_{w|v} |N^{k_w}_{\mathbb{Q}_v}(x)|_v \;=\; 1 \qquad (\text{for } x \in k^\times)$$

because, for K/k an extension of number fields, the *global* norm is the product of the *local* norms:

$$\prod_{w|v} N^{K_w}_{k_v}(x) \;=\; N^K_k(x) \qquad (\text{for } x \in K, \text{ abs value } v \text{ of } k)$$

Corollaries of proof: The *global* degree is the sum of the *local* degrees:

$$\sum_{w|v} [K_w : k_v] = [K : k]$$

The global trace is the sum of the local traces:

$$\mathrm{tr}^K_k(x) \;=\; \mathrm{tr}^{K_w}_{k_v}(x) \qquad (\text{for } x \in K)$$

Classification of completions *(often attributed to Ostrowski) :* The topologically inequivalent (non-discrete) norms on $\mathbb{Q}$ are the usual $\mathbb{R}$ norm and the p-adic $\mathbb{Q}_p$'s.

Proof: Let $| * |$ be a norm on $\mathbb{Q}$. It turns out (intelligibly, if we guess the answer) that the watershed is whether $| * |$ is *bounded* or *unbounded* on $\mathbb{Z}$. That is, the statement of the theorem could be sharpened to say: norms on $\mathbb{Q}$ bounded on $\mathbb{Z}$ are topologically equivalent to p-adic norms, and norms unbounded on $\mathbb{Z}$ are topologically equivalent to the norm from $\mathbb{R}$.

For $| * |$ bounded on $\mathbb{Z}$, in fact $|x| \leq 1$ for $x \in \mathbb{Z}$, since otherwise $|x^n| = |x|^n \to +\infty$ as $n \to +\infty$.

To say that $| * |$ is *bounded* on $\mathbb{Z}$, but *not discrete*, implies $|x| < 1$ for some $x \in \mathbb{Z}$, since otherwise $d(x, y) = |x - y| = 1$ for $x \neq y$, giving the discrete topology.

Then, by unique factorization, $|p| < 1$ for some prime number p. If there were a second prime q with $|q| < 1$, with $a, b \in \mathbb{Z}$ such that $ap^m + bq^n = 1$ for positive integers m, n, then

$$1 \;=\; |1| \;=\; |ap^m + bq^n| \;\leq\; |a| \cdot |p|^m + |b| \cdot |q|^n \;\leq\; |p|^m + |q|^n$$

This is impossible if *both* $|p| < 1$ and $|q| < 1$, by taking m, n large. Thus, for $|*|$ bounded on $\mathbb{Z}$, there is a unique prime p such that $|p| < 1$. Up to normalization, such a norm is the p-adic norm.

Next, claim that if $|a| \leq 1$ for some $1 < a \in \mathbb{Z}$, then $|*|$ is *bounded* on $\mathbb{Z}$. Given $1 < b \in \mathbb{Z}$, write b^n in an a-ary expansion

$$b^n \;=\; c_o + c_1 a + c_2 a^2 + \ldots + c_\ell a^\ell \qquad\qquad (\text{with } 0 \leq c_i < a)$$

and apply the triangle inequality,

$$|b|^n \;\leq\; (\ell + 1) \cdot \underbrace{(1 + \ldots + 1)}_{a} \;\leq\; (n \log_a b + 1) \cdot a$$

Taking n^{th} roots and letting $n \to +\infty$ gives $|b| \leq 1$, and $|*|$ is bounded on $\mathbb{Z}$.

The remaining scenario is $|a| \geq 1$ for $a \in \mathbb{Z}$. For $a > 1$, $b > 1$, the a-ary expansion

$$b^n \;=\; c_o + c_1 a + c_2 a^2 + \ldots + c_\ell a^\ell \qquad\qquad (\text{with } 0 \leq c_i < a)$$

with $|a| \geq 1$ gives

$$|b|^n \;\leq\; (\ell+1)\cdot \underbrace{(1+\ldots+1)}_{a} \cdot |a|^\ell \;\leq\; (n\log_a b + 1)\cdot a \cdot |a|^{n\log_a b + 1}$$

Taking n^{th} roots and letting $n \to +\infty$ gives $|b| \;\leq\; |a|^{\log_a b}$. Similarly, $|a| \;\leq\; |b|^{\log_b a}$. Since $|*|$ is not bounded on $\mathbb{Z}$, there is $C > 1$ such that $|a| = C^{\log |a|}$ for all $0 \neq a \in \mathbb{Z}$. Up to normalization, this is the usual absolute value for $\mathbb{R}$. ///

Remark: A similar argument classifies non-discrete norms on $\mathbb{F}_q(x)$ up to topological equivalence.

Corollary: Up to topological equivalence, every norm on a number field is either $\mathfrak{p}$-adic or arises from $\mathbb{R}$ and $\mathbb{C}$. ///

Remark: Note that the product-formula norms K_w on an extension K of k are *not* the *extensions* of the norm k_v with $w|v$. This is visible on the bottom completion k_v:

$$|x|_w \;=\; |N_{k_v}^{K_w}(x)|_v \;=\; |x^{[K:k]}|_v \;=\; |x|_v^{[K:k]} \qquad\qquad (\text{for } x \in k_v)$$

Indeed, on other occasions, the *extension* is the appropriate object, instead of composing with Galois norm.

Context should clarify what norm is appropriate. Typically, *multiplicative* computations/discussions use the product-formula norm, while genuine *metric* computations/discussions use the *extension*.

Additive (Weak) Approximation: *(Artin-Whaples, Lang)* Let $v_1, \ldots, v_n$ index pairwise topologically inequivalent norms on a field k. Given $x_1, \ldots, x_n \in k$ and $\varepsilon > 0$, there exists $x \in k$ such that

$$|x - x_j|_{v_j} \ < \ \varepsilon \qquad\qquad (\text{for } j = 1, \ldots, n)$$

Remark: When the norms are p-adic, arising from prime ideals in a Dedekind ring $\mathfrak{o}$ inside k, this is Sun-Ze's theorem.

Proof: First, we need to refine the notion of topological inequivalence, to exclude the possibility that the $| * |_1$ topology τ_1 is strictly finer than the $|*|_2$-topology τ_2. This uses the same proof mechanism as the earlier result showing that with two norms giving the *same* topology, each is a power of the other.

Suppose that the identity $(k, \tau_1) \to (k, \tau_2)$ is continuous. Then $|x|_1 < 1$ implies $x^n \to 0$ in the $|*|_1$ topology. Thus, $x^n \to 0$ in the $|*|_2$ topology, so $|x|_2 < 1$. Similarly, if $|x|_1 > 1$, then $|x^{-1}|_1 < 1$, so $|x|_2 > 1$.

Fix y with $|y|_1 > 1$. Given $|x|_1 \geq 1$, there is $t \in \mathbb{R}$ such that $|x|_1 = |y|_1^t$. For rational $a/b > t$, $|x|_1 < |y|_1^{a/b}$, so $|x^b/y^a|_1 < 1$. Then $|x^b/y^a|_2 < 1$, and $|x|_2 < |y|_2^{a/b}$.

Similarly, $|x|_2 > |y|_2^{a/b}$ for $a/b < t$. Thus, $|x|_2 = |y|_2^t$, and

$$|x|_2 \;=\; |y|_2^t \;=\; \left(|y|_1^{\frac{\log|y|_2}{\log|y|_1}}\right)^t \;=\; \left(|y|_1^t\right)^{\frac{\log|y|_2}{\log|y|_1}} \;=\; |x|_1^{\frac{\log|y|_2}{\log|y|_1}} \qquad ///$$

Thus, as a corollary, for $|*|_1$ and $|*|_2$ topologically inequivalent, there exists $x \in k$ with $|x|_1 \geq 1$ and $|x|_2 < 1$.

Similarly, let $|y|_1 < 1$ and $|y|_2 \ge 1$. Then $z = y/x$ has $|z|_1 < 1$ and $|z|_2 > 1$.

Inductively, much as in Sun-Ze's theorem, suppose $|z|_1 > 1$ and $|z|_j < 1$ for $2 \le j \le n$, and find z' such that $|z'|_1 > 1$ and $|z'|_j < 1$ for $2 \le j \le n+1$. Let $|w|_1 > 1$ and $|w|_{n+1} < 1$. There are two cases: for $|z|_{n+1} \le 1$, then $z' = w \cdot z^\ell$ is as desired, for large ℓ. For $|z|_{n+1} > 1$, $z' = w \cdot z^\ell/(1+z^\ell)$ is as desired, for large ℓ.

So there exist $z_1, \ldots, z_n$ with $|z_j| > 1$ while $|z_j|_{j'} \le 1$ for $j' \ne j$. Then $z_j^\ell/(1+z_j^\ell)$ goes to 1 at $|*|_j$, and to 0 in the other topologies. Thus, for large-enough ℓ,

$$x_1 \cdot \frac{z_1^\ell}{1+z_1^\ell} + \ldots + x_n \cdot \frac{z_n^\ell}{1+z_n^\ell} \quad\longrightarrow\quad x_j \qquad \text{(in the } j^{th} \text{ topology)}$$

This proves the (weak) approximation theorem. ///

Recall that the ring of **adeles** $\mathbb{A} = \mathbb{A}_k$ of k is

$$\mathbb{A} \;=\; \mathbb{A}_k \;=\; \mathrm{colim}_S \Big(\prod_{v \in S} k_v \times \prod_{v \notin S} \mathfrak{o}_v \Big)$$

Claim: Imbedding k diagonally in $\mathbb{A}_k$, by

$$\alpha \longrightarrow (\ldots, \alpha, \ldots) \;\in\; \mathbb{A}_k$$

the image of k is *discrete*, and the quotient $\mathbb{A}/k$ is *compact.*

Proof: Recall that a *topological group* is a group with a locally-compact Hausdorff topology in which the group operation and inverse are continuous. (Perhaps counter-intuitively, this disqualifies infinite-dimensional topological vectorspaces!) Usually a topological group will have a *countable basis.*

For *abelian* topological group G and (topologically) *closed* subgroup H, the quotient G/H is a topological group. If H were not closed, the quotient would fail to be Hausdorff.

In topological groups G (as in topological vector spaces), to describe a topology it suffices to give a local basis of neighborhoods at the identity $e \in G$: for all $g \in G$, the map $h \to gh$ is continuous (by definition), and has continuous inverse $h \to g^{-1}h$, so is a homeomorphism. Thus, for basis $\{N_j\}$ of neighborhoods of e, $\{gN_j\}$ is a basis of neighborhoods at g.

A sub*set* Y of a topological space X is *discrete* when every point $y \in Y$ has a neighborhood N in X such that $N \cap Y = \{y\}$.

Claim: A sub*group* Γ of a topological group G is discrete as a sub*set* if and only if the identity e has a neighborhood N in G such that $N \cap \Gamma = \{e\}$.

Proof of Claim: Discreteness certainly implies that e has such a neighborhood. For any other $\gamma \in \Gamma$,

$$\gamma N \cap \Gamma \;=\; \gamma \cdot (N \cap \gamma^{-1}\Gamma) \;=\; \gamma \cdot (N \cap \Gamma) \;=\; \gamma \cdot \{e\} \;=\; \{\gamma\}$$

Thus, every point of Γ is isolated when e is. ///

Claim: A discrete subgroup Γ of G is *closed*.

Note: A discrete sub*set* need not be closed: $\{\frac{1}{n} : 1 \leq n \in \mathbb{Z}\}$ is discrete in $\mathbb{R}$ but is not closed.

Proof of claim: Let N be a neighborhood of e in G meeting Γ just at e. By continuity of the group operation and inversion in G, there is a neighborhood U of e such that $U^{-1} \cdot U \subset N$. Suppose $g \notin \Gamma$ were in the closure of Γ in G. Then gU contains two distinct elements γ, δ of Γ. But

$$\gamma^{-1} \cdot \delta \in (gU)^{-1} \cdot (gU) = N^{-1} \cdot N \subset N$$

contradiction. Thus, Γ is closed in G. ///

Returning to proving k is discrete in $\mathbb{A} = \mathbb{A}_k$, it suffices to find a neighborhood N of $0 \in \mathbb{A}$ meeting k just at 0.

To begin, let

$$N_{\text{fin}} \;=\; \prod_{v|\infty} k_v \times \prod_{v<\infty} \mathfrak{o}_v \;=\; \text{open neighborhood of } 0 \text{ in } \mathbb{A}$$

$N_{\text{fin}} \cap k = \mathfrak{o}$, since requiring local integrality everywhere implies global integrality ($\mathfrak{o}$ is Dedekind). Then it suffices to show that the projection of $\mathfrak{o}$ to $\prod_{v|\infty} k_v = k \otimes_{\mathbb{Q}} \mathbb{R}$ is discrete *there*.

We showed that $\mathfrak{o}$ is a free $\mathbb{Z}$-module of rank $[k:\mathbb{Q}]$, and that a $\mathbb{Z}$-basis $\{e_1, \ldots, e_n\}$ is a $\mathbb{Q}$-basis of k. Because extending scalars preserves free-ness, $\{e_1, \ldots, e_n\}$ is an $\mathbb{R}$-basis of $k \otimes_{\mathbb{Q}} \mathbb{R}$.

This reduces the question to a more classical one: given an $\mathbb{R}$-basis $\{e_1, \ldots, e_n\}$ of an $\mathbb{R}$ vector space V, show that the *lattice* $\Lambda = \bigoplus_j \mathbb{Z}e_j$ is *discrete* in V.

Conveniently, by now we know that a finite-dimensional $\mathbb{R}$-vectorspace has a unique (appropriate) topology, so, by changing coordinates, we can suppose the e_j are the *standard* basis of $\mathbb{R}^n$, so $\Lambda = \mathbb{Z}^n$, and $\mathbb{R}^n$ is given the usual metric topology. Any ball of radius < 1 at 0 meets $\mathbb{Z}^n$ just at 0, proving discreteness.

To show *compactness* of $\mathbb{A}/k$, in a similar fashion: first, show that, given $\alpha \in \mathbb{A}$, there is $x \in k$ such that $\alpha - x \in \prod_{v|\infty} k_v \times \prod_{v<\infty} \mathfrak{o}_v$. Let $0 \neq \ell \in \mathbb{Z}$ such that $\ell\alpha \in \mathfrak{o}_v$ at all $v < \infty$. With $\ell\mathfrak{o} = \prod_j \mathfrak{p}_j^{e_j}$ with $0 < e_j \in \mathbb{Z}$. By Sun-Ze, there is $y \in \mathfrak{o}$ such that $y - \ell\alpha_{\mathfrak{p}_j} \in \mathfrak{p}_j^{e_j} \cdot \mathfrak{o}_{\mathfrak{p}_j}$ for all j. Then $\ell^{-1}y - \alpha$ is locally integral at all finite places, so $x = \ell^{-1}y \in k$ is the desired element.

That is, $\mathbb{A}/k$ has representatives in $\prod_{v|\infty} k_v \times \prod_{v<\infty} \mathfrak{o}_v$. By Tychonoff, the latter is compact.

Again, a $\mathbb{Z}$-basis $\{e_1, \ldots, e_n\}$ of $\mathfrak{o}$ is an $\mathbb{R}$-basis of the real vector space $k_\infty = \prod_{v|\infty} k_v$. Every element of k_∞ has a representative $\sum_j c_j e_j$ with $0 \leq c_j \leq 1$. The collection of such elements is a continuous image (by scalar multiplication and vector addition) of the compact set $[0,1]^n$, so is compact. ///

Remark: Recall that $\mathbb{A}_k/k$ is also the **solenoid** $\lim_{\mathfrak{a}} k_\infty/\mathfrak{a}$, the limit taken over non-zero ideals $\mathfrak{a}$ of $\mathfrak{o}$. This gives another proof of the compactness, again by Tychonoff.