(October 15, 2016)

Examples 02

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[This document is http://www.math.umn.edu/~garrett/m/real/examples_2016-17/real-disc-02.pdf]

[02.1] Prove that every open in \( \mathbb{R}^2 \) is a countable union of Cartesian products \((a, b) \times (c, d)\) of open intervals.

**Discussion:** For a point \( u \) in (non-empty) open \( U \subset \mathbb{R}^2 \), there is an open ball about \( u \) of some positive radius and contained in \( U \). Shrink the ball slightly to have positive rational radius \( r \). Within distance \( r/4 \) of \( u \) there is a point \((x, y) \in \mathbb{Q}^2 \). Then the open ball \( B_u \) of radius \( r/4 \) at \((x, y)\) contains \( u \). Thus, the square \((x - \frac{r}{2}, x + \frac{r}{2}) \times (y - \frac{r}{2}, y + \frac{r}{2})\) contains \( u \). There are only countably-many possibilities for the intervals \((x - \frac{r}{2}, x + \frac{r}{2})\) and \((y - \frac{r}{2}, y + \frac{r}{2})\), so the set of such occurring for \( u \in U \) is countable.

[02.2] Define tent functions of (half-) width \( w \) and height \( h \) centered at 0 by

\[
t_{w,h}(x) = \begin{cases} 
0 & \text{(for } x \leq -w) \\
\frac{h}{w} \cdot (x + w) & \text{(for } -w \leq x \leq 0) \\
h - \frac{h}{w} \cdot x & \text{(for } 0 \leq x \leq w) \\
0 & \text{(for } x \geq w) 
\end{cases}
\]

Show that the functions \( f_n(x) = t_{\frac{1}{n},n}(x - \frac{1}{n}) \), a sequence of narrowing tents just to the right of 0, go to 0 pointwise (everywhere!), but that

\[
\lim_{n} \int_{\mathbb{R}} f_n(x) \cdot g(x) \, dx = g(0) \quad \text{(for all } g \in C^{\infty}(\mathbb{R}))
\]

**Discussion:** Certainly \( f_n(x) = 0 \) for \( x \leq 0 \), for all \( n \). For \( x > 0 \), there is positive integer \( n_0 \) such that \( x > \frac{2}{n_0} \), by the Archimedean property of the reals. Then for \( n \geq n_0 \) the functions \( f_n \) are 0 at \( x \), giving the pointwise convergence to 0.

By design, the integrals of all the \( f_n \) are 1. Given \( \varepsilon > 0 \), let \( \delta > 0 \) be small enough so that \( |g(x) - g(0)| < \varepsilon \) for \( |x-0| < \delta \). For \( n \) large enough so that \( \frac{2}{n} < \delta \),

\[
\left| \int_{\mathbb{R}} f_n(x) \cdot g(x) \, dx - g(0) \right| = \left| \int_{\mathbb{R}} f_n(x) \cdot (g(x) - g(0)) \, dx \right| = \int_{0}^{2/n} f_n(x) \cdot |g(x) - g(0)| \, dx \leq \int_{0}^{2/n} f_n(x) \cdot \varepsilon \, dx = \varepsilon \cdot \int_{0}^{2/n} f_n(x) \, dx = \varepsilon
\]

giving the convergence of the integrals to \( g(0) \) as claimed.

[02.3] Show that the functions \( f_n(x) = t_{\frac{1}{n},n^2}(x - \frac{1}{n}) - t_{\frac{1}{n},n^2}(x + \frac{1}{n}) \), whose graphs are tall tents of area \( n \) upward just to the right of 0, and tall tents downward just to the left of 0, go to 0 pointwise everywhere, but that

\[
\lim_{n} \int_{\mathbb{R}} f_n(x) \cdot g(x) \, dx = 2 \cdot g'(0) \quad \text{(for differentiable } g \text{ with derivative } g' \text{ in } C^{\infty}(\mathbb{R}))
\]

(Thanks to J. Morey for finding an error in the original computation, which lost the factor of 2...)

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Discussion: Certainly $f_n(x) = 0$ for $|x| \geq \frac{\pi}{2}$, for all $n$, and $f_n(0) = 0$ for all $n$. Given $x \neq 0$, there is positive integer $n_o$ such that $|x| > \frac{\pi}{2 n_o}$, by the Archimedean property of the reals. Then for $n \geq n_o$ the functions $f_n$ are 0 at $x$, giving the pointwise convergence to 0.

By a Taylor-Maclaurin expansion, $g(x) = g(0) + g'(0) \cdot x + h(x)$ where $|h(x)/x| \to 0$ as $x \to 0$. By linearity of integrals,

$$\int f_n(x) \cdot g(x) \, dx = \int f_n(x) \cdot g(0) \, dx + \int f_n(x) \cdot g'(0)x \, dx + \int f_n(x) \cdot h(x) \, dx$$

$$= g(0) \int f_n(x) \, dx + g'(0) \int f_n(x) \cdot x \, dx + \int f_n(x) \cdot h(x) \, dx$$

$$= g(0) \cdot 0 + g'(0) \int f_n(x) \cdot x \, dx + \int f_n(x) \cdot h(x) \, dx = g'(0) \int f_n(x) \cdot x \, dx + \int f_n(x) \cdot h(x) \, dx$$

Much as in the previous example, for the integral involving $h$, given $\varepsilon > 0$, let $\delta > 0$ be small enough so that $|h(x)/x| < \varepsilon$ for $0 < |x - 0| < \delta$. For $n$ large enough so that $\frac{\pi}{2 n} < \delta$,

$$\left| \int f_n(x) \cdot h(x) \, dx \right| = \int |f_n(x)| \cdot |h(x)| \, dx = \int_{|x| \leq \frac{\pi}{2 n}} |f_n(x)| \cdot |h(x)| \, dx \leq \int_{|x| \leq \frac{\pi}{2 n}} |f_n(x)| \cdot \varepsilon |x| \, dx$$

$$= \varepsilon \int_{|x| \leq \frac{\pi}{2 n}} |f_n(x)| \cdot |x| \, dx$$

Thus, evaluation of $\int f_n \cdot x$ will also facilitate showing $\int f_n \cdot h \to 0$. Since both $f_n$ and $x$ are odd, their product is even, so

$$\int_{|x| \leq \frac{\pi}{2 n}} f_n(x) \cdot x \, dx = 2 \cdot \int_0^{2/n} f_n(x) \cdot x \, dx = 2 \cdot \int_0^{1/n} n^3 x \cdot x \, dx + 2 \cdot \int_{1/n}^{2/n} (n^2 - n^3 x - \frac{1}{n}) \cdot x \, dx$$

Replacing $x$ by $x + \frac{1}{n}$ in the second integral gives some simplification:

$$2 \cdot \int_0^{1/n} n^3 x \cdot x \, dx + 2 \cdot \int_0^{1/n} (n^2 - n^3 x) \cdot (x + \frac{1}{n}) \, dx = 2 \cdot \int_0^{1/n} n^2 x + (n - n^2 x) \, dx = 2 \int_0^{1/n} n \, dx = 2$$

Thus, $|\int f_n \cdot h| < \varepsilon \cdot 1$ for $n \geq 2/\delta$, so that integral goes to 0, and the coefficient of $g'(0)$ is 2. ///

[02.4] Show that the closed unit ball in $\ell^2$, although closed and bounded, is not compact, by showing it is not sequentially compact.

Discussion: Let $e_n = (0, \ldots, 0, 1, 0, \ldots)$ with the single 1 at the $n^{th}$ place. Then $d(e_m, e_n) = \sqrt{2}$ for $m \neq n$. Thus, the sequence of $e_n$'s has no Cauchy subsequence, so no convergent subsequence. ///

[02.5] Show that the Hilbert cube

$$C = \{(z_1, z_2, \ldots) \in \ell^2 : |z_n| \leq \frac{1}{n}\}$$

is compact. More generally, for any sequence of positive reals $\varepsilon_n$,

$$C(\varepsilon) = \{(z_1, z_2, \ldots) \in \ell^2 : |z_n| \leq \varepsilon_n\}$$

is compact if and only if $\sum_n |\varepsilon_n|^2 < \infty$. 2
Discussion: Probably better to rewrite \( C(\varepsilon) \) as \( C(\delta) \) with \( \delta = (\delta_1, \delta_2, \ldots) \). Use the total boundedness criterion. Given \( \varepsilon > 0 \), by convergence of \( \sum_n \delta_n^2 \), there is \( n_0 \) large enough so that \( \sum_{n \geq n_0} \delta_n^2 < \varepsilon^2 \). The set

\[
C_{n_0} = \{(z_1, z_2, \ldots, z_{n_0}) \in \mathbb{R}^{n_0} : |z_n| \leq \delta_n \}
\]

is a compact subset of \( \mathbb{R}^{n_0} \), so certainly has a finite cover by open balls of radius \( \varepsilon \). Let the centers of these balls be \( w_1, \ldots, w_N \). Let \( j : \mathbb{R}^{n_0} \to \ell^2 \) be the inclusion \( j(z_1, \ldots, z_{n_0}) = (z_1, \ldots, z_{n_0}, 0, 0, \ldots) \). Then we claim that the open balls of radius \( 2\varepsilon \) at \( j(w_1), j(w_2), \ldots, j(w_N) \) cover \( C(\delta) \). Indeed, given \( z = (z_1, z_2, \ldots) \in C(\delta) \), write \( z = j(z') + z'' \) where \( z' = (z_1, \ldots, z_{n_0}) \) and \( z'' = z - j(z') = (0, \ldots, 0, z_{n_0+1}, \ldots) \). There is at least one of the \( w_j \)'s within \( \varepsilon \) of \( z' \): let \( w_{j_0} \) be such. By the triangle inequality for the norm \( | \cdot |_{\ell^2} \) on \( \ell^2, \)

\[
d(z, j(w_{j_0})) = |z - j(w_{j_0})|_{\ell^2} = |j(z') + z'' - j(w_{j_0})|_{\ell^2} \leq |j(z') - j(w_{j_0})|_{\ell^2} + |z''|_{\ell^2} = |z' - w_{j_0}|_{\mathbb{R}^{n_0}} + |z''|_{\ell^2} < \varepsilon + \varepsilon
\]

Thus, \( C(\delta) \) can be covered by finitely-many open balls of radius \( 2\varepsilon \).

\[\text{[02.6] Show that a closed interval } [a, b] \text{ has the expected Lebesgue measure, namely, } |b - a|, \text{ by showing that the inf of } \sum_{j=1}^n |b_n - a_n| \text{ for all finite open covers } [a, b] \subseteq \bigcup_{j=1}^n (a_j, b_j) \text{ is } |b - a|.\]

Discussion: There is no need to use more than one copy of a given interval in approximating the inf, so we can assume the intervals are distinct. Further, we can assume that the finite cover is minimal in the sense that no \( (a_1, b_1) \) is redundant.

Renumbering if necessary, suppose that \( a_1 \leq a_i \) for all \( i = 1, \ldots, n \). Necessarily \( a_1 < a \), or else the left endpoint of \( [a, b] \) is not covered. Without loss of generality, \( b_1 > a \), or else we could have dropped \( (a_1, b_1) \) from the cover. Then there is at least one index \( j > 1 \) such that \( a_j < b_1 \), or else \( (a_1, b_1) \) and \( \bigcup_{j>1} (a_j, b_j) \) are disjoint, so \( [a, b] \cap (a_1, b_1) \) and \( [a, b] \cap \bigcup_{j>1} (a_j, b_j) \) are disjoint and non-empty, contradicting the connectedness of \( [a, b] \). Renumbering if necessary, \( j = 2 \). Then \( b_2 > b_1 \), or else we could have dropped \( (a_2, b_2) \) from the cover. In fact, \( a_2 > a_1 \), or else \( (a_1, b_1) \) is redundant.

Suppose by induction, we have renumbered so that \( a_1 < a_2 < \ldots < a_m, b_1 < \ldots < b_m, a_{i+1} < b_i \) for \( i = 1, \ldots, m - 1, \) and \( a_1 < a \). If \( b_m > b \), we are already done. For \( b_m \leq b \), we claim there is an index \( j > m \) such that \( a_j < b_m \). If not, \( \bigcup_{i = m_a} (a_i, b_i) \) and \( \bigcup_{j > m} (a_j, b_j) \) are disjoint, meeting \( [a, b] \) in non-empty open subsets, contradicting the connectedness of \( [a, b] \). Renumber if necessary so that \( j = m + 1 \). Necessarily \( b_{m+1} > b_m \), or else \( (a_{m+1}, b_{m+1}) \) was redundant. Thus, by induction, we have \( a_1 < a_2 < \ldots < a_n, b_1 < \ldots < b_n, a_{i+1} < b_i \) for \( i = 1, \ldots, m - 1, a_1 < a, \) and \( b_n > b \). Then

\[
\sum_{i=1}^n b_i - a_i = \sum_{i=1}^{n-1} (b_i - a_i) + (b_n - a_n) \geq \sum_{i=1}^{n-1} (a_{i+1} - a_i) + (b_n - a_n)
\]

\[
= (a_2 - a_1) + (a_3 - a_2) + \ldots + (a_n - a_{n-1}) + (b_n - a_n) = b_n - a_1 \geq b - a
\]

as claimed.

\[\text{[02.7] Let } f \text{ be a continuous function on } [0, 1], \text{ with } f(0) = 0 \text{ and } f(1) = 1. \text{ Show that } \{x : f(x) \in [\frac{1}{4}, \frac{3}{4}]\} \text{ has positive Lebesgue measure.}\]

Discussion: By the intermediate value theorem, that set is not empty. It is the inverse image of an open set, so by continuity is open. Thus, it contains a non-empty interval, which has positive measure.

\[\text{[02.8] Analogous to the Cantor middle-thirds set, but with shrinking ratios of what’s removed, form a subset of } [0, 1], \text{ by first removing the middle } 1/4 \text{ of } [0, 1], \text{ then the middle } 1/9 \text{ of the remaining intervals, then the middle } 1/16 \text{ of the remaining intervals, then the middle } 1/25 \text{ of the remaining intervals, } \ldots \text{ Show that the nested intersection of these sets has positive measure.}\]

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(Correction to the original needless specific claim: the measure is $1/2\pi$.)

**Discussion:** At the $n^{th}$ stage, the total of the lengths of the remaining intervals is multiplied by $(1 - \frac{1}{n^2})$. Thus, by the $n^{th}$ stage, the length of the remaining intervals is

$$(1 - \frac{1}{2^2})(1 - \frac{1}{3^2})(1 - \frac{1}{4^2}) \cdots (1 - \frac{1}{n^2})$$

For $0 < x < \frac{1}{2}$, there is a constant $C$ such that $|\log(1 - x)| \leq Cx$. Then the logarithm of the finite product is estimated by

$$\left| \log \left( (1 - \frac{1}{2^2}) \cdots (1 - \frac{1}{n^2}) \right) \right| \leq |\log(1 - \frac{1}{2^2})| + \cdots + |\log(1 - \frac{1}{n^2})| \leq \sum_{m=2}^{n} \frac{1}{m^2} < +\infty$$

Thus, the limit of the finite products is a (finite) non-zero number. //

*[The following is needlessly complicated, as the infinite product actually telescopes. Corrected the constants…]*

If we recall that

$$\prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right) = \frac{\sin \pi z}{\pi z}$$

then

$$\prod_{n=2}^{\infty} \left(1 - \frac{z^2}{n^2}\right) = \frac{\sin \pi z}{1 - z^2 \cdot \pi z}$$

The left-hand side evaluated at $z = 1$ is the desired value. Near $z = 1$ there is the power series expansion

$$\sin \pi z = (\sin \pi) - (\cos \pi) \cdot (z - 1) + \ldots = -(z - 1) + \ldots$$

so (with missing $\pi$ inserted, cancelling the $\pi$ in the denominator)

$$\prod_{n=2}^{\infty} \left(1 - \frac{1}{n^2}\right) = \frac{\sin \pi z}{(1 - z^2) \cdot \pi z} \bigg|_{z=1} = \frac{-(z - 1)\pi + \ldots}{(1 - z) \cdot (1 + z) \cdot \pi z} \bigg|_{z=1} = \frac{\pi + \ldots}{(1 + 1) \cdot \pi} = \frac{1}{2}$$