Examples discussion 04

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[04.1] Comparing $L^p$ spaces. Let $1 \leq p, p' < \infty$. When is $L^p[a,b] \subset L^{p'}[a,b]$ for finite intervals $[a,b]$ and Lebesgue measure? When is $L^p(\mathbb{R}) \subset L^{p'}(\mathbb{R})$? When is $\ell^p \subset \ell^{p'}$?

Discussion: Take $p < p'$. We claim that $L^p[a,b] \supset L^{p'}[a,b]$, with proper containment. The function $f$ that is $(x-a)^{-\frac{1}{p'}}$ on $(a,b]$ and 0 off that interval is not in $L^{p'}$, but is in $L^p$. Given $f \in L^{p'}[a,b]$, let $E$ be the set of $x \in [a,b]$ where $|f(x)| \geq 1$. Then $\int_a^b |f|^{p'} < \infty$ if and only if $\int_E |f|^{p'} < \infty$. On $E$, $|f|^p < |f|^{p'}$, so $\int_E |f|^p < \infty$, and then also $\int_a^b |f|^p < \infty$, so $f \in L^p[a,b]$.

We claim that $L^p(\mathbb{R})$ and $L^{p'}(\mathbb{R})$ are not comparable for $p \neq p'$. Take $1 \leq p < p'$. On one hand, $1/(1+|x|)^{1/p'+\epsilon}$ is in $L^{p'}$ for all $\epsilon > 0$, but not in $L^p$ for $\epsilon$ small enough so that $\frac{1}{p} + \epsilon < \frac{1}{p'}$. On the other hand, the function $f$ that is $x^{-\frac{1}{p'}}$ on $(0,1]$ and 0 off that interval is not in $L^{p'}$, but is in $L^p$.

We claim that for $1 \leq p < p' < \infty$, $\ell^p \subset \ell^{p'}$, with strict containment. Indeed, $f(n) = 1/n^p$ is not in $\ell^{p'}$, but is in $\ell^p$. Let $E = \{n \in \{1,2,\ldots\} : |f(n)| < 1\}$. Then $f \in \ell^p$ if and only if the complement of $E$ is finite, and if $\sum_{n \not\in E} |f(n)|^p < \infty$. Certainly $|f(n)|^p > |f(n)|^p$ for $n \in E$, and the complement of $E$ is finite, so $\sum_{n \not\in E} |f(n)|^p < \sum_{n \in E} |f(n)|^p$, and $f \in \ell^{p'}$.

[04.2] For positive real numbers $w_1, \ldots, w_n$ such that $\sum_i w_i = 1$, and for positive real numbers $a_1, \ldots, a_n$, show that

$$a_1^{w_1} \cdots a_n^{w_n} \leq w_1 \cdot a_1 + \cdots + w_n \cdot a_n$$

Discussion: This is a corollary of Jensen’s inequality, similar to the arithmetic-geometric mean, but with unequal weights. Namely, let $X = \{1,2,\ldots,n\}$ with measure $\mu(i) = w_i$, and function $f(i) = \log a_i$. Then Jensen’s inequality is

$$\exp\left(\sum_{i=1}^n w_i \cdot \log a_i\right) = \sum_{i=1}^n w_i \cdot e^{\log a_i}$$

which simplifies to the assertion. ///

[04.3] In $\ell^2$, show that the point in the closed unit ball closest to a point $v$ not inside that ball is $v/|v|e_2$.

Discussion: The minimum principle assures that there is a unique closest point $w$ in the closed unit ball $B$ to $v$, because $B$ is convex, closed, non-empty, and $v$ is not in $B$.

Suppose $w$ is closer than $v/|v|$. Then

$$|v|^2 - 2|v| + 1 = |v - \frac{v}{|v|}|^2 > |v - w|^2 = |v|^2 - \langle v, w \rangle - \langle w, v \rangle + |w|^2 = |v|^2 - \langle v, w \rangle - \langle w, v \rangle + 1$$

Thus,

$$2|v| < \langle v, w \rangle + \langle w, v \rangle$$

Thus, the sum of the two inner products is positive, and by Cauchy-Schwarz-Bunyakovsky:

$$2|v| < \langle v, w \rangle + \langle w, v \rangle = |\langle v, w \rangle + \langle w, v \rangle| \leq 2|v| \cdot |w|$$

Thus, $1 < |w|$, which is impossible. ///
[04.4] For a measurable set $E \subset [0, 2\pi]$, show that
\[
\lim_{n \to \infty} \int_E \cos nxdx = 0 = \lim_{n \to \infty} \int_E \sin nxdx
\]

**Discussion:** This is an instance of a *Riemann-Lebesgue lemma*, namely, that Fourier coefficients of an $L^2$ function on $[0, 2\pi]$ go to 0. Here, the $L^2$ function is the characteristic function of $E$, and we use sines and cosines instead of exponentials. 

[04.5] One form of the sawtooth function is $f(x) = x - \pi$ on $[0, 2\pi]$. Compute the Fourier coefficients $\hat{f}(n)$. Write out the conclusion of Parseval’s theorem for this function.

**Discussion:** We have the orthonormal basis $e_n(x) = \frac{1}{\sqrt{2\pi}} e^{inx}$ with $n \in \mathbb{Z}$ for the Hilbert space $L^2[0, 2\pi]$. The Fourier coefficients are determined by Fourier’s formula
\[
\hat{f}(n) = \int_0^{2\pi} f(x) \frac{e^{-inx}}{\sqrt{2\pi}} dx
\]

For $n = 0$, this is 0. For $n \neq 0$, integrate by parts, to get
\[
\hat{f}(n) = \left[ f(x) \cdot \frac{e^{-inx}}{\sqrt{2\pi} \cdot (-in)} \right]_0^{2\pi} - \int_0^{2\pi} 1 \cdot \frac{e^{-inx}}{\sqrt{2\pi} \cdot (-in)} dx
\]
\[
= \left( (\pi \cdot \frac{1}{\sqrt{2\pi} \cdot (-in)}) - (-\pi \cdot \frac{1}{\sqrt{2\pi} \cdot (-in)}) \right) - 0 = \frac{2\pi}{\sqrt{2\pi} \cdot (-in)} = \frac{2\pi}{-in}
\]

The $L^2$ norm of $f$ is
\[
\int_0^{2\pi} (x - \pi)^2 dx = \left[ \frac{(x - \pi)^3}{3} \right]_0^{2\pi} = \frac{\pi^3 - (-\pi)^3}{3} = \frac{2\pi^3}{3}
\]

Thus, by Parseval,
\[
\sum_{n \neq 0} \left| \frac{\sqrt{2\pi}}{-in} \right|^2 = \frac{2\pi^3}{3}
\]

This simplifies first to
\[
2 \sum_{n \geq 1} \frac{2\pi}{n^2} = \frac{2\pi^3}{3}
\]

and then to
\[
\sum_{n \geq 1} \frac{1}{n^2} = \frac{\pi^2}{6}
\]

That is, Parseval applied to the sawtooth function evaluates $\zeta(2)$. 

[04.6] For fixed $y \in [0, 2\pi]$, show that there is no $f_y \in L^2[0, 2\pi]$ so that $\langle g, f_y \rangle = g(y)$ for all $g \in L^2[0, 2\pi]$.

**Discussion:** Part of the issue here is whether $L^2$ functions truly have meaningful pointwise values at all, and we generally imagine that they do not, although such a negative fact may be hard to express formulaically.

Among many approaches, one is to suppose such $f$ exists. Choose an orthonormal basis for $L^2[0, 2\pi]$ consisting of the continuous functions $\psi_n(x) = e^{2\pi inx}$, and see what the condition $\langle f_y, \psi_n \rangle = \psi_n(y)$ imposes on the alleged $f_y$. Indeed, this condition completely determines the Fourier coefficients of the alleged $f_y$, so
\[
f_y = \sum_{n \in \mathbb{Z}} \psi_n(y) \cdot \psi_n \quad \text{(with equality in an } L^2 \text{ sense)}
\]
By Parseval,
\[ |f_y|_{L^2}^2 = \sum_n |\psi_n(y)|^2 = +\infty \]
since \(|\psi_n(y)| = 1\) for all \(n\). Thus, there can be no such \(f_y\) in \(L^2\).

[04.7] (In contrast to the previous example’s outcome.) Let \(V\) be the complex vector space of power series \(f(z) = \sum_{n \geq 0} c_n z^n\) convergent on the open unit disk \(D\) in \(C\), having finite norm
\[ |f| = \left( \int_D |f(x + iy)|^2 \, dx \, dy \right)^{\frac{1}{2}} \]
with hermitian inner product
\[ \langle f, g \rangle = \int_D f(x + iy) \cdot \overline{g(x + iy)} \, dx \, dy \]
Show that \(\langle z^m, z^n \rangle = 0\) unless \(m = n\), in which case it is \(\frac{2\pi}{2n + 1}\), and that \(\psi_n(z) = z^n \cdot \sqrt{\frac{2\pi}{2n + 1}}\) is an orthonormal basis for \(V\). Show that the sum \(f_w(z) = \sum_{n \geq 0} \psi_n(z) \overline{\psi_n(w)}\) converges absolutely for \(z, w \in D\), and that
\[ \langle g(-), f_w \rangle = g(w) \quad \text{(for } w \text{ in the disk)} \]
Show that for each fixed \(w \in D\), pointwise evaluation \(g \to g(w)\) is a continuous linear functional on \(V\).