Examples discussion 08

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[This document is http://www.math.umn.edu/~garrett/m/real/examples_2016-17/real-disc-08.pdf]

[08.1] For $f \in L^2(\mathbb{R})$ and $t \in \mathbb{R}$, show that there is a constant $C$ (depending on $f$) such that

$$\left| \int_{t-\delta}^{t+\delta} f(x) \, dx \right| < C \cdot \sqrt{\delta}$$

Formulate and prove the corresponding assertion for $L^p$ with $1 < p < \infty$.

Discussion: Let $h_\delta$ be the characteristic function of $[t-\delta, t+\delta]$. By Cauchy-Schwarz-Bunyakowsky

$$\left| \int_{t-\delta}^{t+\delta} f \right| = |\langle f, h_\delta \rangle_{L^2}| \leq |f|_{L^2} \cdot |h_\delta|_{L^2} = |f|_{L^2} \cdot \sqrt{2\delta}$$

The case of conjugate exponents $\frac{1}{p} + \frac{1}{q} = 1$ is the same, using Hölder’s inequality rather than Cauchy-Schwarz-Bunyakowsky. There is no immediate analogue for $L^1$, although a weaker result is possible, as in the next example. ///

[08.2] For $f \in L^1(\mathbb{R})$ and $t \in \mathbb{R}$, show that, given $\varepsilon > 0$, there is $\delta > 0$ such that

$$\left| \int_{t-\delta}^{t+\delta} f(x) \, dx \right| < \varepsilon$$

Sharpen the first example to show that

$$\int_{t-\delta}^{t+\delta} f(x) \, dx = o(\sqrt{\delta}) \quad (\text{as } \delta \to 0^+)$$

where Landau’s little-o notation is that $f(x) = o(g(x))$ as $x \to a$ when $\lim_{x \to a} f(x)/g(x) = 0$.

Discussion: Let $S_n = \{ x : \frac{1}{n+1} \leq |x-t| < \frac{1}{n} \}$. Then

$$\left| \sum_{n \geq 1} \int_{S_n} f \right| \leq \sum_{n \geq 1} \int_{S_n} |f| \leq |f|_{L^1}$$

Thus, the sum of non-negative terms $\sum_{n \geq 1} \int_{S_n} |f|$ is convergent, so the tails $\sum_{n \geq N} \int_{S_n} |f|$ go to 0 as $N \to +\infty$. Thus,

$$\left| \int_{|x-t| \leq N} f \right| \leq \int_{|x-t| \leq N} |f| = \sum_{n \geq N} \int_{S_n} |f|$$

goes to 0 as $N \to +\infty$. Then this idea can be applied to $\int_{|x-t| < \delta} |f|^p$ in the previous example. ///

[08.3] Compute $e^{-\pi x^2} * e^{-\pi x^2}$ and $\frac{\sin x}{x} * \frac{\sin x}{x}$. (Be careful what you assert: $\frac{\sin x}{x}$ is not in $L^1(\mathbb{R})$.)

Discussion: The idea is to invoke $f * g = (\hat{f} \cdot \hat{g})$ for even functions $f, g \in L^1$, since for even functions the inverse Fourier transform is the same as the forward Fourier transform. Conveniently, Gaussians are in $L^1 \cap L^2$, and have Fourier transforms which are again Gaussians:

$$e^{-\pi a x^2}(\xi) = \frac{1}{\sqrt{a}} e^{-\pi \xi^2/a} \quad (\text{for } a > 0)$$
so
\[ e^{-\pi x^2} \cdot e^{-\pi y^2}(\xi) = e^{-\pi x^2} \cdot e^{-\pi y^2}(\xi) = e^{-2\pi x^2}(\xi) = \frac{1}{\sqrt{2}} e^{-\pi \xi^2/2} \]

For the other example, the bound \(|f \ast g|_{L^1} \leq |f|_{L^p} \cdot |g|_{L^q}\) for conjugate exponents \(p, q\) shows that \(f \ast g \in L^1\) for \(f, g \in L^2\). Thus, the same identity holds for \(f, g \in L^2\), with the Plancherel extension of Fourier transform. That is, \(\hat{f}\) and \(\hat{g}\) need not be the literal integrals for the Fourier transform, but its extension by continuity to \(L^2\). With \(\chi_a\) the characteristic function of \([-a, a]\), recall that
\[
\hat{\chi}_a(\xi) = \int_{-a}^a e^{-2\pi i x \xi} dx = \left[ e^{-2\pi i x \xi} \right]_{-a}^a = e^{-2\pi i a \xi} - e^{2\pi i a \xi} \over -2\pi i \xi = \frac{\sin 2\pi a \xi}{\pi \xi}
\]

Thus,
\[
(\pi \cdot \chi_{a/2\pi})^\wedge(\xi) = \frac{\sin \xi}{\xi}
\]

Then
\[
\left(\frac{\sin x}{x} \ast \frac{\sin x}{x}\right)(\xi) = \left((\pi \cdot \chi_{a/2\pi}) \ast (\pi \cdot \chi_{a/2\pi})\right)^\wedge(\xi) = \pi \cdot (\pi \cdot \chi_{a/2\pi})^\wedge(\xi) = \pi \cdot \frac{\sin \xi}{\xi}
\]

[08.4] Let \(K(x, y) \in L^2([a, b] \times [a, b])\), and attempt to define a map \(T : L^2[a, b] \to L^2[a, b]\) by
\[
Tf(x) = \int_a^b K(x, y) f(y) dy
\]

Show that \(Tf\) is well-defined a.e. as a pointwise-valued function. Show that \(T\) really does map \(L^2\) to itself by showing that
\[
|Tf|_{L^2[a, b]} \leq |K|_{L^2([a, b] \times [a, b])} \cdot |f|_{L^2[a, b]}
\]

(One would say that \(K(, )\) is a \textit{Schwartz kernel} for the map \(T\). Yes, this use is in conflict with the use of \textit{kernel} of a map to refer to things that map to 0.) In the previous situation, show that the Hilbert-space adjoint \(T^*\) of \(T\) has Schwartz kernel \(\hat{K}(y, x)\).

**Discussion:** By Fubini-Tonelli, \(y \to K(x, y)\) is measurable for almost all \(x\), so \(Tf(x)\) is defined almost everywhere (assuming convergence of the integral). By Cauchy-Schwarz-Bunyakowsky, and Fubini-Tonelli as needed,
\[
\int_a^b |Tf(x)|^2 dx = \int_a^b \left| \int_a^b K(x, y) f(y) dy \right|^2 dx \leq \int_a^b \int_a^b |K(x, y)|^2 dy \cdot \int_a^b |f(y')|^2 dy' dx
\]
\[
= |f|_{L^2}^2 \cdot \int_a^b \int_a^b |K(x, y)|^2 dx dy = |f|_{L^2[a, b]}^2 \cdot |K|_{L^2([a, b] \times [a, b])}^2 < +\infty
\]

Thus, \(T\) is \textit{bounded}, so is a \textit{continuous} linear map of \(L^2[a, b]\) to itself. ///

[08.5] The Volterra operator \(Vf(x) = \int_0^x f(y) dy\) on \(L^2[0, 1]\) has kernel

\[
K(x, y) = \begin{cases} 
1 & \text{for } 0 \leq y \leq x \leq 1 \\
0 & \text{for } 0 \leq x < y \leq 1 
\end{cases}
\]

Determine the (Schwartz) kernel for \(T = V \circ V^*\). Find some eigenfunctions for \(T\). (Recall that \(V\) has no eigenfunctions!) (Hint: apply \(d/dx\) to the equation \(Tf = \lambda \cdot f\) and presume that the differentiation passes inside the integral.)
Discussion: The kernel $K^*(x, y)$ for an adjoint $V^*$ is always obtained by the procedure $\overline{K(y, x)}$. Here, the kernel is real-valued. For two operators $S, T$ with respective kernels $K_S(x, y)$ and $K_T(x, y)$, using Fubini-Tonelli as necessary,

$$(S \circ T)f(x) = (S(Tf))(x) = \int_0^1 K_S(x,t)Tf(t) \, dt = \int_0^1 K_S(x,t)\left( \int_0^1 K_T(t,y) \, f(y) \, dy \right) \, dt$$

$$= \int_0^1 \left( \int_0^1 K_S(x,t)K_T(t,y) \, dt \right) f(y) \, dy$$

so the kernel of the composite is

$$K_{S \circ T}(x,y) = \int_0^1 K_S(x,t)K_T(t,y) \, dt$$

Thus, the kernel $L(x,y)$ for $V \circ V^*$ is

$$K_{V \circ V^*}(x,y) = \int_0^1 K_V(x,t)K_V(y,t) \, dt = \int_0^1 \begin{cases} 1 & \text{(for } 0 \leq t \leq x \leq 1) \\ 0 & \text{(for } 0 \leq x < t \leq 1) \\ 0 & \text{(for } 0 \leq y < t \leq 1) \end{cases} dt$$

$$= \int_{t \leq x, t \leq y} 1 \, dt = \min(x,y)$$

$$\int_0^1 \min(x,y) \, f(y) \, dy = \int_0^x y \, f(y) \, dy + \int_x^1 x \, f(y) \, dy = \int_0^x y \, f(y) \, dy + x \int_x^1 \, f(y) \, dy$$

For $f \in L^2$, Cauchy-Schwarz-Bunyakowsky shows that the latter expression side is continuous as a function of $x$. Thus, an eigenfunction equation $\lambda \cdot f = (V \circ V^*)f$ for $f \in L^2$ and $\lambda \neq 0$ implies that $f$ is continuous. Then from

$$\lambda \cdot f(x) = \int_0^x y \, f(y) \, dy + x \int_x^1 \, f(y) \, dy$$

the fundamental theorem of calculus implies that $f \in C^1$. By induction on $k$, $f \in C^k$ for all $k$, so $f$ is smooth. Differentiating the latter expression,

$$\lambda \cdot f'(x) = x \cdot f(x) + \int_x^1 f(y) \, dy - x \cdot f(x) = \int_x^1 \, f(y) \, dy$$

Differentiating again, $\lambda \cdot f'' = -f$. Thus, $f(x) = Ae^{cx} + Be^{-cx}$ for some constants $A, B$, with $c = \sqrt{-\lambda}$. But this is only a necessary condition for an eigenfunction, not sufficient.

One way to determine allowable $\lambda$ is to directly compute the integral

$$\int_0^1 \min(x,y) \cdot (Ae^{cy} + Be^{-cy}) \, dy$$

and examine the condition that this be equal to $-1/c^2 \cdot (Ae^{cx} + Be^{-cx})$. This would involve two integrations by parts. Equivalently, but somewhat more lightly, for each fixed $x \in [0, 1],$

$$F(y) = \begin{cases} 0 & \text{(for } y < 0) \\ y & \text{(for } 0 \leq y < x) \\ x & \text{(for } x \leq y < 1) \\ 0 & \text{(for } y \geq 1) \end{cases}$$
We might recognize this as being closely related to the Dirac comb $\delta_x$ for almost all $x$. For this to be such, its derivative is (locally) $1$ away from $x$, its derivative is $0$ at $x$, and has a $\delta_n$ at each $n \in \mathbb{Z}$. That is, $c \in \pi i + 2\pi i \mathbb{Z}$ with corresponding eigenfunction $e^{cx} - e^{-cx}$.

\[ [08.6] \text{The sawtooth function is first defined on } [0, 1) \text{ by } \sigma(x) = x - \frac{1}{2}, \text{ and then extended to } \mathbb{R} \text{ by periodicity so that } \sigma(x + n) = \sigma(x) \text{ for all } x \in \mathbb{R} \text{ and } n \in \mathbb{Z}. \text{ Determine its Fourier expansion. Describe the derivative } \sigma' \text{ of } \sigma. \]

**Discussion:** The $0^\text{th}$ Fourier coefficient is $0$. For $n \neq 0$, integrating by parts once, the $n^\text{th}$ Fourier is $-1/2\pi in$. That is, at least converging in $L^2$,

\[ \sigma(x) = \sum_{n \neq 0} \frac{1}{2\pi in} e^{2\pi inx} \]

In fact, from the Fourier-coefficient criterion for Sobolev spaces, $\sigma \in H^{1/2-\varepsilon}$ for all $\varepsilon > 0$. Differentiating termwise (in an extended sense),

\[ \sigma' = -\sum_{n \neq 0} e^{2\pi inx} \quad \text{(convergent in } H^{1/2-\varepsilon}) \]

We might recognize this as being closely related to the Dirac comb

\[ \delta_x = \sum_{n \in \mathbb{Z}} e^{2\pi inx} \quad \text{(convergent in } H^{1/2-\varepsilon}) \]

Specifically, $\sigma' = 1 - \delta_x$. Also, looking at the description of $\sigma$ directly, its derivative is (locally) $1$ away from $\mathbb{Z}$, and has a $-\delta_n$ for all $n \in \mathbb{Z}$. That is, yet again,

\[ \sigma' = 1 - \sum_{n \in \mathbb{Z}} \delta_n = 1 - \delta_x \]

\[ [08.7] \text{Given } f \text{ in the Schwartz space } \mathcal{S}, \text{ show that there is } F \in \mathcal{S} \text{ with } F' = f \text{ if and only if } \int_{\mathbb{R}} f = 0. \]

**Discussion:** On one hand, if $f = F'$ for $F \in \mathcal{S}$, then $\int_{-\infty}^{x} f(y) \, dy = F(x)$. Since $\lim_{x \to +\infty} F(x) = 0$, $\int_{\mathbb{R}} f = 0$.

On the other hand, if $\int_{\mathbb{R}} f = 0$, let $F(x) = \int_{-\infty}^{x} f$, and show that $F \in \mathcal{S}$. Since $F' = f$ by the fundamental theorem of calculus, the (higher) derivatives of $F$ are those of $f$, so all that needs to be shown is that $F$ itself is of rapid decay. For $x \to -\infty$,

\[ |F(x)| \leq \int_{-\infty}^{x} |f| \leq \int_{-\infty}^{x} |y|^{-N} \sup_{t \in \mathbb{R}} |y \cdot f(t)| \, dy \leq \sup_{t \in \mathbb{R}} |y \cdot f(t)| \cdot \int_{-\infty}^{x} |y|^{-N} \, dy = \sup_{t \in \mathbb{R}} |y \cdot f(t)| \cdot \frac{|x|^{1-N}}{N-1} \]
Let $\delta$ be any test function that is non-negative real-valued and $\infty$-weakly decreasing. But it might be better to check that, viewing the apparent integral as an extension-by-continuity of literal integrals, with test functions (or Schwartz functions) $f$ approaching $\delta$, one can readily check that $\delta \in \mathcal{E}'$, by letting $K$ be any compact containing $x_0$, and then as for continuity on $\mathcal{S}'$,

$$|\delta_{x_0}(f) - \delta_{x_0}(g)| \leq \sup_{x \in K} |(f - g)(x)|$$

Thus, given $\varepsilon > 0$, for $\sup_{x \in \mathbb{R}} |(f - g)(x)| < \delta = \varepsilon$, we have $|\delta_{x_0}(f) - \delta_{x_0}(g)| < \varepsilon$.

To compute the Fourier transform of $\delta_{x_0}$, it is entirely reasonable to imagine that the following produces the correct outcome:

$$\hat{\delta}_{x_0}(\xi) = \int_{\mathbb{R}} \delta_{x_0}(x) e^{-2\pi i \xi x} \, dx = \delta_{x_0}(x \to e^{-2\pi i \xi x}) = e^{-2\pi i \xi x_0}$$

And, indeed, this conclusion is correct, and this is the effective way to think about and perform such computations. However, we do also want to understand why and how such a computation can be understood as being entirely rigorous. For example, the indicated integral cannot be a literal integral, since $\delta_{x_0}$ is not a pointwise-valued (classical) function. Likewise, although $\delta_{x_0} \in \mathcal{S}'$, the exponential function is not in $\mathcal{S}$, but only in $\mathcal{E} = C^\infty(\mathbb{R})$. Using the duality characterization of $\mathcal{E}'$, one can readily check that $\delta_{x_0} \in \mathcal{E}'$, by letting $K$ be any compact containing $x_0$, and then as for continuity on $\mathcal{S}'$,

$$|\delta_{x_0}(f) - \delta_{x_0}(g)| \leq \sup_{x \in K} |(f - g)(x)|$$

But it might be better to check that, viewing the apparent integral as an extension-by-continuity of literal integrals, with test functions (or Schwartz functions) $f_n$ approaching $\delta_{x_0}$ in the $\mathcal{S}'$ topology,

$$(\mathcal{S}' - \lim_n f_n) = \mathcal{S}' - \lim_n \hat{f}_n = \mathcal{S}' - \lim_n \int_{\mathbb{R}} f_n(x) e^{-2\pi i \xi x} \, dx = e^{-2\pi i \xi x_0}$$

The $\mathcal{S}'$-continuity of the Fourier transform assures that the outcome is independent of the choice of $\{f_n\}$. Let $\varphi$ be any test function that is non-negative real-valued and $\int_{\mathbb{R}} \varphi = 1$. Let $f_n(x) = \frac{1}{n} \varphi(n(x - x_0))$. This is a sort of approximate identity concentrated at $x_0$. By continuity of $x \to e^{-2\pi i \xi x}$ (and monotone convergence or less), we have the natural pointwise estimate

$$\lim_n \int_{\mathbb{R}} f_n(x) e^{-2\pi i \xi x} = e^{-2\pi i \xi x_0} \quad \text{(for fixed } \xi \in \mathbb{R})$$

But this pointwise limit is not quite the $\mathcal{S}'$ limit we want. Rather, we want to show that, for every $\eta \in \mathcal{S}$,

$$\lim_n \int_{\mathbb{R}} \eta(\xi) \cdot \left( \int_{\mathbb{R}} f_n(x) e^{-2\pi i \xi x} \, dx \right) \, d\xi = \int_{\mathbb{R}} \eta(\xi) \cdot e^{-2\pi i \xi x_0} \, d\xi$$
And, unsurprisingly, via Fubini-Tonelli,
\[
\int_{\mathbb{R}} \eta(\xi) \cdot \left( \int_{\mathbb{R}} f_n(x) e^{-2\pi i \xi x} \, dx \right) \, d\xi = \int_{\mathbb{R}} f_n(x) \int_{\mathbb{R}} \eta(\xi) e^{-2\pi i \xi x} \, d\xi \, dx = \int_{\mathbb{R}} f_n(x) \hat{\eta}(x) \, dx \rightarrow \hat{\eta}(x_0)
\]
by the continuity of \( \hat{\eta} \in \mathcal{S} \).

In fact, the same argument justifies, for once and for all, the computation of Fourier transforms of \( u \in \mathcal{E}^* \subset \mathcal{S}^* \) by \( \hat{u}(\xi) = u(x) \to e^{-2\pi i \xi x} \).

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[08.9] Let \( u(x) = e^x \cdot \sin(e^x) \). Explain in what sense the integral \( \int_{\mathbb{R}} f(x) u(x) \, dx \) converges for every \( f \in \mathcal{S} \).

**Discussion:** The idea is to integrate by parts, noting that \( u = v' \) with \( v(x) = \cos(e^x) \). We must be careful with the boundary terms:

\[
\int_{\mathbb{R}} f(x) u(x) \, dx = \int_{\mathbb{R}} f(x) v'(x) \, dx = \lim_{M,N \to +\infty} \int_{-M}^{N} f(x) v'(x) \, dx
\]

\[
= \lim_{M,N \to +\infty} \left( \left[ f(x) v(x) \right]_{-M}^{N} - \int_{-M}^{N} f'(x) v(x) \, dx \right)
\]

Since \( v(x) \) is bounded and \( f' \) is of rapid decay, the limit **exists**, so the original integral is convergent. Further, the value is correctly determined by integration by parts, namely

\[
- \int_{-\infty}^{\infty} f'(x) v(x) \, dx = - \int_{-\infty}^{\infty} f'(x) \cos(e^x) \, dx
\]

That is, for \( f \in \mathcal{S} \) and functions such as \( u \) obtained by differentiating bounded smooth functions, integration by parts is completely justifiable via the natural estimates.  

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