(March 24, 2017)

**Examples discussion 10**

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[This document is http://www.math.umn.edu/~garrett/m/real/notes_2016-17/real-disc-10.pdf]

[10.1] With \( g(x) = f(x + x_o) \), express \( \hat{g} \) in terms of \( \hat{f} \), first for \( f \in \mathcal{S}(\mathbb{R}^n) \), then for \( f \in \mathcal{S}(\mathbb{R}^n)^* \).

**Discussion:** For \( f \in \mathcal{S}(\mathbb{R}^n) \), the literal integral computes the Fourier transform:

\[
\hat{g}(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i \xi \cdot x} g(x) \, dx = \int_{\mathbb{R}^n} e^{-2\pi i \xi \cdot x} f(x + x_o) \, dx
\]

Replacing \( x \) by \( x - x_o \) in the integral gives

\[
\hat{g}(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i \xi \cdot (x-x_o)} f(x) \, dx = e^{2\pi i \xi \cdot x_o} \int_{\mathbb{R}^n} e^{-2\pi i \xi \cdot x} f(x) \, dx = e^{2\pi i \xi \cdot x_o} \cdot \hat{f}(\xi)
\]

The precise corresponding statement for tempered distributions cannot refer to pointwise values. Write \( \psi_{x_o} \) for the function \( \xi \to e^{2\pi i \xi \cdot x_o} \). Since \( \psi_{x_o} \) is bounded, for a tempered distribution \( u \), \( \psi_{x_o} \cdot u \) is the tempered distribution described by

\[
(\psi_{x_o} \cdot u)(\varphi) = u(\psi_{x_o} \cdot \varphi) \quad \text{(for } \varphi \in \mathcal{S})
\]

This is compatible with multiplication of (integrate-against-) functions \( \mathcal{S} \subset \mathcal{S}^* \). Also, let translation \( u \to T_{x_o} u \) be defined by \( (T_{x_o} u)(\varphi) = u(T_{-x_o} \varphi) \), again compatibly with integration against Schwartz functions. In these terms, the above argument shows that

\[
(T_{x_o} f)(\varphi) = \psi_{x_o} \cdot \hat{f} \quad \text{(for } f \in \mathcal{S})
\]

This formulation avoids reference to pointwise values, and thus could make sense for tempered distributions.

One argument is **extension by continuity:** Fourier transform is a continuous map \( \mathcal{S}^* \to \mathcal{S}^* \), as is translation \( u \to T_{x_o} u \), so the identity extends by continuity to all tempered distributions.  

Another argument is by **duality:** first,

\[
(T_{x_o} u)(\varphi) = (T_{x_o} u)(\varphi) = u(T_{-x_o} \varphi) = u((\psi_{x_o} \cdot \varphi)\cdot)
\]

by applying the identity to \( \varphi, \hat{\varphi} \in \mathcal{S} \). Going back, this is

\[
\hat{u}(\psi_{x_o} \cdot \varphi) = (\psi_{x_o} \cdot \hat{u})(\varphi) \quad \text{(for all } \varphi \in \mathcal{S})
\]

Altogether, \( (T_{x_o} u)(\varphi) = \psi_{x_o} \cdot \hat{u} \).

[10.2] Compute \( \hat{\cos x} \).

**Discussion:** Start from \( \hat{\delta} = 1 \). Using the previous example’s identity,

\[
(T_{x_o} \delta)(\varphi) = \psi_{x_o} \cdot 1 = \psi_{x_o}
\]

By Fourier inversion, \( \psi_{x_o} = T_{-x_o} \delta \). Thus,

\[
\hat{\cos x} = \frac{1}{2}(\hat{\cos 1/2} + \hat{\cos -1/2}) = \frac{1}{2}(\hat{T_{-1/2} \delta} + \hat{T_{1/2} \delta})
\]

Written in terms of mock-pointwise-values, this is \( \hat{\cos}(\xi) = \frac{\delta(\xi - \frac{1}{2\pi}) + \delta(\xi + \frac{1}{2\pi})}{2} \).
[10.3] Smooth functions $f \in \mathcal{E}$ act on distributions $u \in \mathcal{D}(\mathbb{R})^*$ by a dualized form of pointwise multiplication: $(f \cdot u)(\varphi) = u(f \varphi)$ for $\varphi \in \mathcal{D}(\mathbb{R})$. Show that if $x \cdot u = 0$, then $u$ is supported at 0, in the sense that for $\varphi \in \mathcal{D}$ with spt $\varphi \neq 0$, necessarily $u(\varphi) = 0$. Thus, by the theorem classifying such distributions, $u$ is a linear combination of $\delta$ and its derivatives. Show that in fact $x \cdot u = 0$ implies that $u$ is a multiple of $\delta$ itself.

Discussion: For $\varphi \in \mathcal{D}$ whose support does not include 0, the function $1/x$ is defined and smooth on spt $\varphi$. Thus, $x \to \varphi(x)/x$ is in $\mathcal{D}$. For such $\varphi$,

$$u(\varphi) = u(x \cdot \frac{\varphi}{x}) = 0$$

Thus, spt $u = \{0\}$, so is a finite linear combination $u = \sum_{i=0}^{\infty} c_i \delta^{(i)}$ with scalars $c_i$. To see that in fact only $\delta$ itself can appear, we use the idea that $\delta = e^\pi i x$ is in $\mathcal{D}$ around 0. Then $\eta \cdot x^i \in \mathcal{D}$, and

$$\delta^{(i)}(\eta \cdot \frac{x^j}{j!}) = \begin{cases} 1 & \text{(for } i = j) \\ 0 & \text{(for } i \neq j) \end{cases}$$

In particular, this shows that the derivatives of $\delta$ are linearly independent. For $0 \leq j \in \mathbb{Z}$,

$$0 = (x \cdot u)(x^j) = (x \cdot \sum_i c_i \delta^{(i)})(x^j) = \sum_i c_i \delta^{(i)}(x \cdot x^j) = \sum_i c_i \delta^{(i)}(x^{j+1}) = (j + 1)! \cdot c_{j+1}$$

Thus, $c_j = 0$ for $j \geq 1$, and $u$ is a multiple of $\delta$ itself. ///

[10.4] Show that the principal value functional $u(\varphi) = P.V. \int_{\mathbb{R}} \frac{\varphi(x)}{x} \, dx$ satisfies $x \cdot u = 1$.

Discussion: For $\varphi \in \mathcal{D}$,

$$u(\varphi) = \lim_{\varepsilon \to 0^+} \int_{|x| \geq \varepsilon} \frac{x \cdot \varphi(x)}{x} \, dx = \lim_{\varepsilon \to 0^+} \int_{|x| \geq \varepsilon} \varphi(x) \, dx = \int_{\mathbb{R}} \varphi(x) \, dx = \int_{\mathbb{R}} 1 \cdot \varphi(x) \, dx = 1(\varphi)$$

since $\varphi$ is continuous at 0. Thus, $x \cdot u = 1$. ///

[10.5] Compute the Fourier transform of the sign function

$$\text{sgn}(x) = \begin{cases} 1 & \text{(for } x > 0) \\ -1 & \text{(for } x < 0) \end{cases}$$

Hint: $\frac{d}{dx} \text{sgn} = 2\delta$. Since Fourier transform converts $d/dx$ to multiplication by $2\pi ix$, this implies that $(2\pi i)x \cdot \hat{\text{sgn}} = 2\delta = 2$. Thus, $(\pi i)x \cdot \hat{\text{sgn}} = 1$.

Discussion: From the hint, $x \cdot (\pi i \hat{\text{sgn}}) = 1$. Also, the principal-value functional $u$ from the previous example satisfies $x \cdot u = 1$. Thus,

$$x \cdot (u - \pi i \hat{\text{sgn}}) = 0$$

By another earlier example, this implies that $u - (\pi i \hat{\text{sgn}})$ is a multiple of $\delta$. In fact, the multiple is 0, because $\delta$ is even, while $u$, sgn, and thus $\hat{\text{sgn}}$, are all odd. [1] That is, $\hat{\text{sgn}} = \frac{1}{\pi i} u$. ///

[1] This notion of parity can be defined for distributions from the obvious notion for functions $(\theta \cdot f)(x) = f(-x)$, and then $(\theta \cdot v)(f) = v(\theta \cdot f)$ for distributions $v$.  

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[0.1] **Remark:** In particular, it is *not quite* that $\widehat{\text{sgn}}(\xi) = 1/\pi i \xi$. Indeed, $1/\xi$ is *not* locally integrable, so does not directly describe a distribution. This example shows that, yes, $\xi \cdot \text{sgn} = 1/\pi i$, but apparently we cannot just *divide* (pointwise values). Indeed, we have proven that the principal-value integral is the Fourier transform (up to constants), and it is not quite just an integral.

[10.6] Compute the Fourier transform of $|x|$.

**Discussion:** From $\frac{d}{dx} |x| = \text{sgn} x$, taking Fourier transforms,

$$\hat{\text{sgn}} = \left( \frac{d}{dx} |x| \right) = 2\pi i \cdot \hat{x}$$

Recall that in the previous example it was just barely *not ok* to divide by $\xi$, and the principal-value functional was not quite a literal integral against $1/x$. Similarly, but even more so, here we *cannot* just divide through by $\xi$ to obtain $|x|$ from the principal-value functional.

Similarly, from $(\frac{d}{dx})^2 |x| = 2\delta$, by Fourier transform, $(2\pi i)^2 \cdot \hat{\xi}^2 \cdot \hat{|x|} = 2 \cdot 1 = 2$ and $-2\pi^2 \cdot \hat{\xi}^2 \cdot \hat{|x|} = 1$, but we can’t just divide.

We can try to make a $1/x^2$ version of the earlier principal-value functional, such as

$$u(\varphi) = \lim_{\varepsilon \to 0^+} \int_{|x| \geq \varepsilon} \varphi(x) - \varphi(0) \frac{dx}{x^2}$$

In fact, we can see that this $u$ is the (distributional) derivative of the previous principal-value functional: integrating by parts,

$$\int_{|x| \geq \varepsilon} \frac{\varphi(x) - \varphi(0)}{x^2} \, dx = \left[ \frac{\varphi(x) - \varphi(0)}{-x} \right]_{-\varepsilon}^{\infty} + \left[ \frac{\varphi(x) - \varphi(0)}{-x} \right]_{-\infty}^{-\varepsilon} - \int_{|x| \geq \varepsilon} \frac{\varphi'(x)}{-x} \, dx$$

$$= - \frac{\varphi(\varepsilon) - \varphi(0)}{-\varepsilon} + \frac{\varphi(-\varepsilon) - \varphi(0)}{-(-\varepsilon)} + \int_{|x| \geq \varepsilon} \frac{\varphi'(x)}{x} \, dx = \frac{\varphi(\varepsilon) - \varphi(0)}{\varepsilon} - \frac{\varphi(-\varepsilon) - \varphi(0)}{-\varepsilon} + \int_{|x| \geq \varepsilon} \frac{\varphi'(x)}{x} \, dx$$

In the limit, the first two terms give $\varphi'(0) - \varphi'(0) = 0$. Thus, this principal-value functional $u$ is the distributional derivative of the earlier one.

As in the earlier example, we claim that $x^2 \cdot u = 1$: for $\varphi \in \mathcal{D}$,

$$(x^2 \cdot u)(\varphi) = u(x^2 \cdot \varphi) = \lim_{\varepsilon \to 0^+} \int_{|x| \geq \varepsilon} x^2 \cdot \varphi(x) - 2 \cdot \varphi(0) \frac{dx}{x^2} = \lim_{\varepsilon \to 0^+} \int_{|x| \geq \varepsilon} \varphi(x) \, dx = \int_{\mathbb{R}} \varphi(x) \, dx = 1(\varphi)$$

Thus, both $x^2 \cdot (u - 2\pi^2 \hat{|x|}) = 1$ and $x^2 \cdot u = 1$. Thus, $x^2 \cdot (u - 2\pi^2 \hat{|x|}) = 0$. As in an earlier example, this implies that $u - 2\pi^2 \hat{|x|} = a \cdot \delta + b \cdot \delta'$ for some scalars $a, b$. Since $u, |x|$ and, hence, $\hat{|x|}$ are *even*, in fact that difference must be a multiple of $\delta$, since $\delta'$ is *odd*.

To determine the constant, it suffices to apply both functionals to a convenient $\varphi \in \mathcal{S}$, such as $\varphi(x) = e^{-\pi x^2}$, which is its own Fourier transform. On one hand,

$$u(\varphi) = \lim_{\varepsilon \to 0^+} \int_{|x| \geq \varepsilon} \frac{e^{-\pi x^2}}{x} \, dx = \lim_{\varepsilon \to 0^+} \int_{|x| \geq \varepsilon} \frac{-2\pi x e^{-\pi x^2}}{x} \, dx$$

$$= \lim_{\varepsilon \to 0^+} \int_{|x| \geq \varepsilon} -2\pi e^{-\pi x^2} \, dx = \int_{\mathbb{R}} -2\pi e^{-\pi x^2} \, dx = -2\pi$$
On the other hand,
\[
|x|(e^{-\pi x^2}) = |x|(\widehat{e^{-\pi x^2}}) = |x|(e^{-\pi x^2}) = \int \Re |x| \cdot e^{-\pi x^2} \, dx = 2 \int_0^\infty x e^{-\pi x^2} \, dx = \int_0^\infty e^{-\pi x} \, dx = \frac{1}{\pi}
\]
by replacing \(x\) by \(\sqrt{x}\). Thus,
\[
a = a \delta(e^{-\pi x^2}) = (u - 2\pi^2 |x|)(e^{-\pi x^2}) = (-2\pi) - 2\pi^2 \cdot \left(\frac{1}{\pi}\right) = -2\pi - 2\pi = -4\pi
\]
That is,
\[
\widehat{|x|} = \frac{u}{2\pi^2} + 4\delta
\]
... that is, if no constants got lost.

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[0.2] **Remark:** Again, the principal-value functional \(u\) cannot be a literal integral.

[10.7] Determine the Schwartz kernel \(K(\cdot,\cdot)\) for the identity map \(\mathcal{D}(\mathbb{T}^n) \to \mathcal{D}(\mathbb{T}^n)\), and show that it is in \(H^{-\frac{2}{\epsilon},-\frac{\epsilon}{2}}(\mathbb{T}^{2n})\) for every \(\epsilon > 0\).

**Discussion:** Let \(T\) be the identity map \(\mathcal{D}(\mathbb{T}^n) \to \mathcal{D}(\mathbb{T}^n)\) viewed as a map \(\mathcal{D}(\mathbb{T}^n) \to \mathcal{D}(\mathbb{T}^n)^*\) via the natural imbedding \(\mathcal{D} \subset \mathcal{D}^*\). Write \(\psi_\xi\) for the function \(\psi_\xi(x) = e^{2\pi i \xi \cdot x}\) for \(\xi \in \mathbb{Z}^n\) and \(x \in \mathbb{R}/\mathbb{Z}\). Anticipating that \(K\) is at worst in \(H^{-\infty}(\mathbb{T}^{2n})\), we can write a Fourier expansion \(K = \sum_{\xi,\eta \in \mathbb{Z}^n} c_{\xi,\eta} \psi_\xi \otimes \psi_\eta\) with coefficients to be determined. [2] Of course there is no reason to think that this converges **pointwise**, and this doesn’t matter. The Schwartz kernel for \(T: \mathcal{D} \to \mathcal{D}^*\) is characterized by
\[
K(\varphi \otimes Tf) = (Tf)(\varphi) \quad \text{(for all } \varphi \in \mathcal{D})
\]
Applying this to \(\varphi = \psi_\alpha\) and \(f = \psi_\beta\),
\[
c_{\alpha,\beta} = K(\psi_\alpha, \psi_\beta) = (T\psi_\beta)(\psi_\alpha) = \int_{\mathbb{T}^n} \psi_\beta \cdot \psi_\alpha = \begin{cases} 0 & \text{(for } \beta \neq -\alpha \in \mathbb{Z}^n) \\ 1 & \text{(for } \beta = \alpha \in \mathbb{Z}^n) \end{cases}
\]
The latter **necessary** condition already completely determines \(K\): apparently \(K = \sum_{\alpha} \psi_\alpha \otimes \psi_{-\alpha}\). However, we should give a reason why this expression really does give the identity map on \(\mathcal{D}(\mathbb{T}^n)\). Certainly
\[
\left| \sum_{\alpha \in \mathbb{Z}^n} \psi_\alpha \otimes \psi_{-\alpha} \right|_{H^s}^2 = \sum_{\alpha \in \mathbb{Z}^n} |1|^2 \cdot (1 + |\alpha|^2)^s
\]
is finite if and only if \(s < -\frac{n}{2}\). Thus, for every \(\epsilon > 0\), \(K \in H^{-\frac{s}{2},-\epsilon}(\mathbb{T}^{2n}) \subset H^{-\infty}(\mathbb{T}^{2n}) = H^{\infty}(\mathbb{T}^{2n})^*\). That is, that Fourier expansion converges in a Sobolev space and does give a distribution on \(\mathbb{T}^{2n}\).

Since finite linear combinations of \(\psi_\alpha\) are **dense** in \(\mathcal{D}(\mathbb{T}^n)\), and since \(K\) is continuous on \(H^{\infty}(\mathbb{T}^n) \otimes H^{\infty}(\mathbb{T}^n) \subset H^{\infty}(\mathbb{T}^{2n})\), the earlier computation of \(K(\psi_\alpha \otimes \psi_\beta)\) extends by continuity to certify that \(K(f \otimes g) = \int f \cdot g\) for \(f, g \in \mathcal{D}(\mathbb{T}^n)\).

///

[2] The tensor notation here is just a way to refer to the function \(x, y \rightarrow \psi_\xi(x) \cdot \psi_\eta(y)\) without using arguments.