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Examples discussion 11

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[This document is http://www.math.umn.edu/~garrett/m/real/examples_examples_discussion_11.pdf]

[11.1] For $T : V \to V$ a continuous (=bounded) linear map of a Banach space $V$ to itself, show that the operator norm is an upper bound for absolute values of all eigenvalues $\lambda$: $|\lambda|_C \leq |T|_{\text{op}}$. Further, show that $|T|_{\text{op}}$ is an upper bound for all of the spectrum, that is, $T - \lambda$ is invertible for $|\lambda|_C > |T|_{\text{op}}$.

Discussion: First, for $Tv = \lambda \cdot v$ for $0 \neq v \in V$, without loss of generality take $|v| = 1$, and then

$$|T|_{\text{op}} = \sup_{|w| \leq 1} |Tw| \geq |Tv| = |\lambda| \cdot |v| = |\lambda|$$

Second, for $|\lambda|_C > |T|_{\text{op}}$, we have $|T/\lambda|_{\text{op}} < 1$, so

$$S = 1 + T/\lambda + (T/\lambda)^2 + \ldots = \lim_N (1 + T/\lambda + (T/\lambda)^2 + \ldots + (T/\lambda)^N)$$

is convergent in the operator norm on the Banach space of continuous/bounded linear operators on $V$, since the tails go to 0. Since $1 - T/\lambda$ is continuous, as expected

$$(1 - T/\lambda) \cdot S = S \cdot (1 - T/\lambda) = \lim_N (1 - T/\lambda) \cdot (1 + T/\lambda + (T/\lambda)^2 + \ldots + (T/\lambda)^N) = \lim_N 1 - (T/\lambda)^{N+1} = 1$$

since $(T/\lambda)^N \to 0$: if there were any doubt,

$$||(T/\lambda)^N||_{\text{op}} \leq |T/\lambda|^N \to 0$$

since $|T/\lambda|_{\text{op}} < 1$. Thus, the inverse $S = (1 - T/\lambda)^{-1}$ exists (as a continuous linear operator), and $\lambda$ is not in any part of the spectrum.

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[11.2] (Approximate eigenvectors and continuous spectrum) Let $T : V \to V$ be a continuous linear operator on a Hilbert space $V$. For $\lambda \in \mathbb{C}$, a sequence $\{v_n\}$ of vectors (normalized so that all their lengths are 1 or at least bounded away from 0) such that $(T - \lambda)v_n \to 0$ as $n \to +\infty$ is an approximate eigenvector for $\lambda$. Show that for $\lambda$ not an eigenvalue for $T$, $\lambda$ has an approximate eigenvector if and only if $\lambda$ is in the spectrum of $T$.

Discussion: (Sometimes this is called Weyl’s criterion.) In fact, this criterion is not reliable for detecting some types of residual spectrum. [1] We give an example at the end of the discussion. Certainly if $\lambda$ is an eigenvector, with non-zero eigenvalue $v$, the constant sequence $v, v, v, \ldots$ fits the requirement.

For general spectrum, let $S = T - \lambda$. For $v_1, v_2, \ldots$ with $|v_n| = 1$ and $Sv_n \to 0$, any alleged (continuous[2]) $S^{-1}$ would give, interchanging $S^{-1}$ and the limit by continuity,

$$0 = S^{-1}(\lim_n Sv_n) = \lim_n S^{-1}Sv_n = \lim_n v_n$$

contradiction. Thus, existence of an approximate eigenvector for $T - \lambda$ implies that $T - \lambda$ is not invertible.

[1] Recall that residual spectrum of $T$ is $\lambda$ such that $T - \lambda$ is injective, but does not have dense image.

[2] Recall that when there is an everywhere-defined, linear inverse $S^{-1}$ to $S$, necessarily $S$ is a continuous bijection, and by the open mapping theorem $S$ is open. That is, there is $\delta > 0$ such that $|Sv| \geq \delta \cdot |v|$ for all $v$. This exactly asserts the boundedness of $S^{-1}$, so $S^{-1}$ is continuous.
Conversely, for \( S = T - \lambda \) not invertible, but \( \lambda \) not an eigenvector, then \( S \) is **injective** but not **surjective**. We do need a further assumption: suppose that the image of \( S \) is **not closed**. In that case, \( S \) is injective, not surjective, and by non-closedness of the image there is \( v_o \) (with \(|v_o| = 1\)) not in the image of \( S \), and \( v_1, v_2, \ldots \) such that \( Sv_1, Sv_2, \ldots \to v_o \). Since \(|Sv_n| \leq |S|_{op} \cdot |v_n| \) and \(|Sv_n| \to |v_o| = 1 \), there is a uniform non-zero lower bound on the lengths \(|v_n|\), and these form an approximate eigenvector for \( 0 \) for \( S = T - \lambda \), hence for \( \lambda \) for \( T \).

As noted, the case that \( \lambda \) is not an eigenvector, \( T - \lambda \) is not surjective, and/or the image of \( T - \lambda \) is **closed**, can only occur for non-normal \( T \). For example, \( T : \ell^2 \to \ell^2 \) by

\[
T(c_1, c_2, \ldots) = (c_1, 0, c_2, 0, c_3, 0, \ldots)
\]

is injective, not surjective, and has closed image. It is not invertible, but there is no approximate eigenvector for \( 0 \), so the criterion fails in this (non-normal) example. \(///\)

**Discussion:** This operator is self-adjoint, since \( \sin x \) is real-valued:

\[
\langle Tf, g \rangle = \int_{\mathbb{R}} f(x) \cdot \sin x \cdot \overline{g(x)} \, dx = \int_{\mathbb{R}} f(x) \cdot \overline{g(x)} \cdot \sin x \, dx = \langle f, Tg \rangle
\]

For a function \( f \) and fixed \( \lambda \in \mathbb{C} \) such that \( f(x) \cdot \sin x = \lambda \cdot f(x) \) for almost all \( x \), for \( x \) such that \( f(x) \neq 0 \), necessarily \( \sin x = \lambda \). Since \( \sin x \) assumes any particular value at most countably many times, \( f = 0 \) almost everywhere. Thus, there are no eigenvalues.

Since \( T \) is self-adjoint, it is normal, so there is no residual spectrum. Thus, Weyl’s criterion via approximate eigenvectors suffices to determine the remainder of the spectrum, which will be **continuous**. Given a value \( \lambda \in [-1, 1] \), let \( x_o \in \mathbb{R} \) be such that \( \sin x_o = \lambda \). We claim that an approximate eigenvector for \( \lambda \) can be formed by functions concentrated ever-more-closely at \( x_o \), such as

\[
v_n(x) = \begin{cases} \sqrt{n} & (\text{for } |x - x_o| \leq \frac{1}{2n}) \\ 0 & (\text{otherwise}) \end{cases}
\]

By design, \(|v_n| = 1\). Since \( \sin x \) is continuous, given \( \varepsilon > 0 \) there is \( \delta > 0 \) such that \(|\sin x - \sin x_o| < \varepsilon \) for \(|x - x_o| < \delta \). For \( n \) large enough so that \( 1/2n < \delta \),

\[
|Tv_n - \lambda v_n|_{L^2}^2 = |v_n \cdot \sin x - \lambda \cdot v_n|_{L^2}^2 = \int_{x_o - \frac{1}{2n}}^{x_o + \frac{1}{2n}} n \cdot |\sin x - \sin x_o|^2 \, dx < \int_{x_o - \frac{1}{2n}}^{x_o + \frac{1}{2n}} n \cdot \varepsilon^2 \, dx = \varepsilon^2
\]

Thus, \( Tv_n - \lambda v_n \to 0 \), and \( \{v_n\} \) is an approximate identity for \( \lambda \), so every \( \lambda \in [-1, 1] \) is in the continuous spectrum. \(///\)

**11.3** Show that the multiplication operator \( T : L^2(\mathbb{R}) \to L^2(\mathbb{R}) \) by \( Tf(x) = f(x) \cdot \sin x \) has empty discrete spectrum. Show that it is self-adjoint. Show that \( T \) has continuous spectrum the interval \([-1, 1] \). (We know that self-adjoint (or merely **normal**) operators have only point spectrum and continuous spectrum, that is, no left-over **residual** spectrum.)

**Discussion:** For a function \( f \) and fixed \( \lambda \in \mathbb{C} \) such that \( f(x) \cdot \sin x = \lambda \cdot f(x) \) for almost all \( x \), for \( x \) such that \( f(x) \neq 0 \), necessarily \( \sin x = \lambda \). Since \( \sin x \) assumes any particular value at most countably many times, \( f = 0 \) almost everywhere. Thus, there are no eigenvalues.

Since \( T \) is self-adjoint, it is normal, so there is no residual spectrum. Thus, Weyl’s criterion via approximate eigenvectors suffices to determine the remainder of the spectrum, which will be **continuous**. Given a value \( \lambda \in [-1, 1] \), let \( x_o \in \mathbb{R} \) be such that \( \sin x_o = \lambda \). We claim that an approximate eigenvector for \( \lambda \) can be formed by functions concentrated ever-more-closely at \( x_o \), such as

\[
v_n(x) = \begin{cases} \sqrt{n} & (\text{for } |x - x_o| \leq \frac{1}{2n}) \\ 0 & (\text{otherwise}) \end{cases}
\]

By design, \(|v_n| = 1\). Since \( \sin x \) is continuous, given \( \varepsilon > 0 \) there is \( \delta > 0 \) such that \(|\sin x - \sin x_o| < \varepsilon \) for \(|x - x_o| < \delta \). For \( n \) large enough so that \( 1/2n < \delta \),

\[
|Tv_n - \lambda v_n|_{L^2}^2 = |v_n \cdot \sin x - \lambda \cdot v_n|_{L^2}^2 = \int_{x_o - \frac{1}{2n}}^{x_o + \frac{1}{2n}} n \cdot |\sin x - \sin x_o|^2 \, dx < \int_{x_o - \frac{1}{2n}}^{x_o + \frac{1}{2n}} n \cdot \varepsilon^2 \, dx = \varepsilon^2
\]

Thus, \( Tv_n - \lambda v_n \to 0 \), and \( \{v_n\} \) is an approximate identity for \( \lambda \), so every \( \lambda \in [-1, 1] \) is in the continuous spectrum. \(///\)

**11.4** Let \( r_1, r_2, r_3, \ldots \) be an enumeration of the rational numbers inside the interval \([0,1]\). Define \( T : \ell^2 \to \ell^2 \) by \( T(c_1, c_2, \ldots) = (r_1 c_1, r_2 c_2, \ldots) \). Show that \( T \) is a continuous/bounded linear operator, is self-adjoint, has eigenvalues exactly the \( r_1, r_2, \ldots \), and continuous spectrum the whole interval \([0,1]\) (with rationals removed, if one insists on disjointness of discrete and continuous spectrum).

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3 The image is not closed, for example, when \( T \) (hence \( S \)) has no residual spectrum, which is the case when \( T \) (hence \( S \)) is **normal**.
Discussion: Since the set \( \{|r_1|, |r_2|, \ldots \} \) is bounded by 1, the operator norm of \( T \) is at most 1, so it is bounded, hence continuous. Since the \( r_n \) are all real, the operator is self-adjoint:

\[
\langle T(a_1, a_2, \ldots), (b_1, b_2, \ldots) \rangle = \langle (r_1 a_1, r_2 a_2, \ldots), (b_1, b_2, \ldots) \rangle = \sum_n r_n a_n \cdot \overline{b_n}
\]

\[
= \sum_n a_n \cdot \overline{r_n b_n} = \langle (a_1, a_2, \ldots), T(b_1, b_2, \ldots) \rangle
\]

When \( \lambda \cdot (c_1, c_2, \ldots) = T(c_1, c_2, \ldots) = (r_1 c_1, r_2 c_2, \ldots) \), necessarily \( \lambda \cdot c_n = r_n \cdot c_n \) for all \( n \). When \( c_n \neq 0 \), this implies \( \lambda = r_n \). Since the \( r_n \) are distinct, there can be (at most) one index \( n \) for which \( c_n \neq 0 \), and then \( \lambda = r_n \). Conversely, every \( r_n \) is obviously an eigenvalue.

Since we know that the whole spectrum is closed in \( \mathbb{C} \), it contains at least the closure of the rationals in \([0, 1]\), namely, \([0, 1]\) itself. Since \( T \) is self-adjoint, its spectrum is contained in \( \mathbb{R} \). [4] Since the spectrum is bounded by \( |T|_{op} = 1 \), it is contained in \([-1, 1]\).

To see that \( \lambda \in [-1, 0) \) is not in the spectrum, in that \( (T - \lambda)(c_1, c_2, \ldots) = ((r_1 - \lambda)c_1, (r_2 - \lambda)c_2, \ldots) \), we have \( |r_n - \lambda| \geq |\lambda| > 0 \), so the inverse \((T - \lambda)^{-1}\) can be written down immediately: \((T - \lambda)^{-1}(c_1, c_2, \ldots) = ((r_1 - \lambda)^{-1}c_1, (r_2 - \lambda)^{-1}c_2, \ldots) \) and there is a uniform upper bound \( |(r_n - \lambda)^{-1}| \leq |\lambda|^{-1} \). Finally, given irrational \( \lambda \in [0, 1] \), let \( r_{n_1}, r_{n_2}, \ldots \) be rationals such that \( r_{n_i} \to \lambda \). With standard basis \( \{e_n\} \) for \( \ell^2 \), we claim that \( \{e_n\} \) is an approximate eigenvector for \( \lambda \): given \( \varepsilon > 0 \), let \( N \) be sufficiently large so that \( |r_{n_i} - \lambda| < \varepsilon \) for \( i \geq N \). For \( n \geq N \),

\[
|\langle (T - \lambda)e_n, e_n \rangle| = |(r_{n_i} - \lambda)e_n| = |r_{n_i} - \lambda||e_n| = |r_{n_i} - \lambda| < \varepsilon
\]

Thus, indeed, \( (T - \lambda)e_n \to 0 \), and the \( e_n \) give an approximate identity for \( \lambda \), so \( \lambda \) is in the spectrum.

\[
\text{Discussion: Let } e_1, e_2, \ldots \text{ be the standard (Hilbert-space) basis for } \ell^2 \text{. If the } r_n \text{ do not go to 0, then there is a subsequence } r_{n_1}, r_{n_2}, \ldots \text{ bounded away from 0. Since } T \text{ is compact, the images } Te_{n_i} = r_{n_i}e_{n_i} \text{ must have a convergent subsequence. But } |r_{n_i}e_{n_i} - r_{n_j}e_{n_j}|^2 = |r_{n_i}|^2 + |r_{n_j}|^2 \text{ for } i \neq j \text{, and this is bounded away from 0, so there is no convergent subsequence, contradicting the compactness of } T \text{. Thus, in fact, } r_n \to 0.
\]

For the converse, perhaps the most economical approach is to observe that \( T \) is an operator-norm limit of finite-rank operators, hence compact:

\[
T_n(c_1, c_2, \ldots, c_n, c_{n+1}, \ldots) = (c_1, c_2, \ldots, c_n, 0, 0, \ldots)
\]

[4] The proof that self-adjoint operators \( T \) have spectrum inside \( \mathbb{R} \) has more content than just the analogous assertion about eigenvectors. For \( Tv = \lambda v \) with \( v \neq 0 \), of course

\[
\lambda(v, v) = \langle \lambda v, v \rangle = \langle Tv, v \rangle = \langle v, Tv \rangle = \langle v, \lambda v \rangle = \overline{\lambda(v, v)}
\]

shows that any eigenvalues are real. Since self-adjoint operators have no residual spectrum, to find the rest of the spectrum it suffices to identify approximate eigenvectors. Note that for self-adjoint \( T \) always \( \langle Tv, v \rangle = \overline{\langle v, Tv \rangle} = \langle T^*v, v \rangle = \overline{\langle v, Tv \rangle} \), so \( (Tv, v) \) is real. Then for \( (T - \lambda)v_n \to 0 \), certainly \( \langle (T - \lambda)v_n, v_n \rangle \to 0 \), so the imaginary parts go to 0. These are

\[
\text{Im}(\langle (T - \lambda)v_n, v_n \rangle) = \text{Im}(Tv_n, v_n) + \text{Im}(\lambda \cdot \langle v_n, v_n \rangle) = 0 + \text{Im}(\lambda \cdot \langle v_n, v_n \rangle)
\]

Since \( |v_n| \) are bounded away from 0, there can be an approximate identity only for \( \lambda \in \mathbb{R} \).

[5] For such a simple operator, a similar device shows that \( \lambda \notin \mathbb{R} \) is not in the spectrum.
The estimate on the operator norms is
\[
|T - T_n|_{\text{op}} = \sup_{|v| \leq 1} |(0, \ldots, 0, r_{n+1} v_{n+1}, \ldots)| = \sup_{k \geq n} |r_k| \to 0
\]

Less efficiently, we can refer to definitions, and use the total boundedness criterion for compact closure. Given \( \varepsilon > 0 \), let \( N \) be large enough so that \( |r_n| < \varepsilon \) for \( n \geq N \). Write \( v = (v_1, v_2, \ldots) \in \ell^2 \) as
\[
v = (v_1, \ldots, v_N, 0, 0, \ldots) + (0, \ldots, 0, v_{N+1}, \ldots, v_{N+2}, \ldots)
\]

Let \( B' \) be the intersection of the unit ball \( B \subset \ell^2 \) with the copy of \( \mathbb{C}^N \subset \ell^2 \) with non-zero components only at the first \( N \) places. Let \( B'' \) be the intersection of \( B \) with the subspace of \( \ell^2 \) with 0 entries at the first \( N \) places. Certainly \( B' + B'' \supset B \) and \( B' \perp B'' \).

By design, \( |Tv''| \leq \varepsilon \) for \( v'' \in B'' \). Since \( TB' \) is a bounded subset of a finite-dimensional space \( \mathbb{C}^N \), it has compact closure, so is totally bounded, so can be covered by finitely-many \( \varepsilon \)-balls \( U_1, \ldots, U_k \). Then \( TB \subset TB' + TB'' \subset (U_1 + TB'') \cup \ldots \cup (U_k + TB'') \), and every \( U_i + TB'' \) is contained in a \( 2\varepsilon \)-ball. Thus, \( TB \) is totally bounded, hence, has compact closure. ///

[11.6] Let \( V \) be the Volterra operator \( Vf(x) = \int_0^x f(t) \, dt \) on \( L^2[0,1] \). Show that \( |V^n|_{\text{op}} \to 0 \) as \( n \to +\infty \). Show that the spectrum of \( V \) is just \( \{0\} \).

Discussion: In fact, \( |V^n|_{\text{op}} \to 0 \) is a weaker conclusion than was intended, in part because it would not help showing that the spectrum of \( V \) is just \( \{0\} \). Rather we would want to show something like \( \lim_{n \to \infty} |V^n|_{\text{op}} = 0 \).

Nevertheless, one way to estimate the behavior of \( V^n \) is to show that \( |V|_{\text{op}} < 1 \). For the latter, recall that a linear map \( T : L^2(X) \to L^2(X) \) given by \( K(, \cdot) \in L^2(X \times X) \) by
\[
Tf(x) = \int_X K(x,y) f(y) \, dy
\]
is a Hilbert-Schmidt operator, with operator norm bounded by the \( L^2(X \times X) \) norm of \( K(, \cdot) \) (this is the Hilbert-Schmidt norm of the operator). Thus,
\[
|V|_{\text{op}}^2 \leq \int_0^1 \int_0^1 \left\{ \begin{array}{ll} 1 & \text{(for } y < x) \\ 0 & \text{(for } y > x) \end{array} \right\}^2 \, dx \, dy = \int_0^1 \int_0^x 1 \, dx \, dy = \int_0^1 x \, dx = \frac{1}{2} < 1
\]

Thus, certainly, \( |V^n|_{\text{op}} \leq |V|_{\text{op}} \to 0 \). ///

We also recall the argument that \( V \) has no eigenvalues: when \( \int_0^x f(t) \, dt = \lambda \cdot f(x) \) for \( \lambda \neq 0 \), application of Cauchy-Schwarz-Bunyakovsky to the left-hand side shows that \( f \) is continuous. Then the equation shows that \( v \in C^1 \) (and induction shows that \( f \in C^\infty \)). Differentiating the eigenfunction condition, \( f = \lambda \cdot f' \). We know how to solve this equation: all solutions are multiples of \( x \to e^{x/\lambda} \). However, the eigenfunction relation also shows that \( f(0) = 0 \) (and \( f \) is continuous, so this holds in a strong sense: we are not allowed to change its values on sets of measure 0), which does not hold for any of these functions. Thus, there are no eigenvalues. ///

Now we return to demonstration of the stronger result, that \( |V^n|_{\text{op}}^{1/n} \to 0 \). To do so, we consider the integral/Schwartz kernel for the iterate \( V^n \):
\[
(V^n f)(x) = \int_0^x \int_0^{x_{n-1}} \ldots \int_0^{x_2} \int_0^{x_1} f(y) \, dy \, dx_1 \, dx_2 \ldots \, dx_{n-2} \, dx_{n-1}
\]
Changing the order of integration to isolate the kernel, this is
\[
(V^n f)(x) = \int_0^x f(y) \left( \int_y^x \int_y^{x_{n-1}} \ldots \int_y^{x_2} 1 \, dx_1 \, dx_2 \ldots dx_{n-2} \, dx_{n-1} \right) \, dy
\]
By induction, the inner integral is \(\frac{(x-y)^{n-1}}{(n-1)!}\) for \(0 \leq y \leq x\). That is, the kernel \(K_n(x,y)\) for \(V^n\) is
\[
K_n(x,y) = \begin{cases} 
\frac{(x-y)^{n-1}}{(n-1)!} & \text{(for } y < x) \\
0 & \text{(for } y > x) 
\end{cases}
\]
and its \(L^2\) norm squared is
\[
\int_0^1 \int_0^x |K_n(x,y)|^2 \, dy \, dx = \int_0^1 \int_0^x \frac{(x-y)^{2(n-1)}}{((n-1)!)^2} \, dy \, dx
\]
\[
= \int_0^1 \frac{x^{2n-1}}{((n-1)!)^2 (2n-1)} \, dx = \frac{1}{((n-1)!)^2 (2n-1) (2n)} < \left(\frac{1}{n!}\right)^2
\]
Thus, its \(L^2\) norm is bounded by \(1/n!\). Since \([n!]^{1/n} \to +\infty\), the spectral radius is 0. \(///\)

To show that \(V - \lambda\) is invertible for all \(\lambda \neq 0\), it is not sufficient to know that \(|V|_{op} < 1\) nor that \(|V^n|_{op} \to 0\), but knowing \(|V^n|_{op}^{1/n} \to 0\) does suffice, as follows.

To show \(T - \lambda = -\lambda \circ (1 - T/\lambda)\) is invertible for \(\lambda \neq 0\), it suffices to show that the series
\[
1 + T/\lambda + (T/\lambda)^2 + (T/\lambda)^3 + \ldots
\]
for the obvious candidate for \((1 - T/\lambda)^{-1}\) converges in operator norm. That is, we want
\[
1 + |T|_{op} + |(T/\lambda)^2|_{op} + |(T/\lambda)^3|_{op} + \ldots < +\infty
\]
The left-hand side is
\[
1 + |T|_{op} \cdot |\lambda|^{-1} + |(T|_{op})^2 \cdot |\lambda|^{-2} + |(T|_{op})^3 \cdot |\lambda|^{-3} + \ldots
\]
Applying the root test,
\[
\limsup_n \left( |T|_{op} \cdot |\lambda|^{-n} \right)^{1/n} \leq \limsup_n \left( |T|_{op}^{1/n} \cdot |\lambda|^{-1} \right) = |\lambda|^{-1} \cdot \limsup_n (|T|_{op})^{1/n} \leq |\lambda|^{-1} \cdot 0 < 1
\]
so the series converges for all \(\lambda \neq 0\). That is, \((T - \lambda)^{-1}\) exists for all \(\lambda \neq 0\). \(///\)

[11.1] Remark: For self-adjoint (or, more generally, normal) operators \(T\), in fact \(\lim_n |T^n|_{op}^{1/n} = |T|_{op}\). However, as we see for the Volterra operator, in general \(\lim_n |T^n|_{op}^{1/n} \leq |T|_{op}\). For not-necessarily-normal operators, the spectral radius is \(\limsup_n |V^n|_{op}^{1/n}\). The argument given for the Volterra operator shows that in general the spectral radius is an upper bound for the spectrum.

[11.2] Remark: In fact, the spectral radius is a sharp bound for the absolute values \(|\lambda|\) for \(\lambda\) in the spectrum. [... iou ...]

[6] That \((n!)^{1/n} \to +\infty\) certainly follows from Stirling’s asymptotic, but also more elementary considerations. For example, for each fixed \(1 \leq k \in \mathbb{Z}\), for \(n \geq k\)
\[
\left(\frac{1}{n!}\right)^{1/n} \leq (k^{n-k+1})^{1/n} = k^{1 - \frac{k-1}{n}} \to k
\]
Since this holds for every fixed \(k\), the limit must be \(+\infty\).