Examples discussion 12

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[This document is http://www.math.umn.edu/~garrett/m/real/examples_2016-17/real-disc-12.pdf]

[12.1] Let $T$ be a compact operator $T : V \to W$ for Hilbert spaces $V, W$. For $S$ a continuous/bounded operator on $V$, show that $T \circ S : V \to W$ is compact. For $R$ a continuous/bounded operator on $W$, show that $R \circ T : V \to W$ is compact.

Discussion: For $T \circ S$, the image of the unit ball under $S$ is contained in some ball $c \cdot B$, where $B$ is the unit ball, because $S$ is bounded. Since $T$ is linear, $T(c \cdot B) = c \cdot TB$. Since $TB$ is pre-compact, its continuous image under multiplication by $c$ is also pre-compact. Proof: for $c = 0$, we’re done. For $c > 0$, given a finite cover of $TB$ by balls $w_i + B_x$ where $B_x$ is the ball of radius $\varepsilon > 0$ centered at 0. The images $c \cdot (w_i + B_x) = cw_i + cB_x$ cover $c \cdot TB$, and have radius $c \cdot \varepsilon$. Replacing $\varepsilon$ by $\varepsilon/c$ gives balls of radius $\varepsilon$ covering $c \cdot TB$.

For $R \circ T$, similarly as in the previous case, given a finite cover of $TB$ by balls $w_i + B_x$ of radius $\varepsilon > 0$, the images $R(w_i + B_x) = Rw_i + RB_x$ are contained in balls $Rw_i + cB_x$, where $c = |R|_{\text{op}}$ will suffice.

[12.2] (Rellich’s lemma on the circle) For $s < t \in \mathbb{R}$, show that the inclusion map $H^t(\mathbb{T}) \to H^s(\mathbb{T})$ is compact. (Hint: Use the orthogonal bases $\psi_n(x) = e^{2\pi inx}$, and note that their lengths in $H^s(\mathbb{T})$ vary depending on $s$. Thus, if we choose isomorphisms of $H^s$ and $H^t$ to $l^2(\mathbb{Z})$, the inclusion $H^t \to H^s$ sending $\psi_n \to \psi_n$ will not be the identity map on those copies of $l^2(\mathbb{Z})$.)

Discussion: The $H^s$-norm of $\psi_n$ is $(1 + n^2)^s$, so $\psi_n/(1 + n^2)^s$ is an orthonormal basis. The identity map sends

$$\frac{\psi_n}{(1 + n^2)^t} \to \frac{\psi_n}{(1 + n^2)^t} = \frac{\psi_n}{(1 + n^2)^s} \cdot (1 + n^2)^{s-t}$$

Thus, viewed as a map of $l^2(\mathbb{Z})$ to itself, it multiplies the $n^{th}$ element of an orthonormal basis by $(1 + n^2)^{s-t}$. For $t > s$, these go to 0, and we know that this implies that the map is compact.

[12.3] Let $K(.)$ be a measurable function on $\mathbb{R}^2$, with a bound $B$ such that $\int_{\mathbb{R}} |K(x,y)| \, dx \leq B$ for every $y$, and $\int_{\mathbb{R}} |K(x,y)| \, dy \leq B$ for every $x$. Show that $Tf(x) = \int_{\mathbb{R}} K(x,y) f(y) \, dy$ gives a continuous linear map $L^p \to L^p$ for every $1 < p < \infty$, with $|Tf|_{L^p} \leq B \cdot |f|_{L^p}$. (Hint: Hölder’s inequality.)

Discussion: There is a slight further algebraic trick beyond just Hölder’s inequality, manifest in the following: letting $q$ be the dual exponent so that $\frac{1}{p} + \frac{1}{q} = 1$,

$$|Tf(x)| = \left| \int_{\mathbb{R}} K(x,y) f(y) \, dy \right| \leq \int_{\mathbb{R}} |K(x,y)|^1 \cdot |f(y)| \, dy = \int_{\mathbb{R}} |K(x,y)|^{\frac{1}{q}} \cdot |K(x,y)|^{\frac{1}{p}} \cdot |f(y)| \, dy$$

$$\leq \left( \int_{\mathbb{R}} |K(x,y)|^{\frac{1}{q}} \, dy \right)^{1/q} \cdot \left( \int_{\mathbb{R}} |f(y)|^{p} \, dy \right)^{1/p}$$

$$= \left( \int_{\mathbb{R}} |K(x,y)| \, dy \right)^{1/q} \cdot \left( \int_{\mathbb{R}} |K(x,y)| \cdot |f(y)|^{p} \, dy \right)^{1/p} \leq B^{1/q} \cdot \left( \int_{\mathbb{R}} |K(x,y)| \cdot |f(y)|^{p} \, dy \right)^{1/p}$$

invoking Hölder. Using this and invoking Fubini-Tonelli,

$$|Tf|_{L^p}^p = \int_{\mathbb{R}} \left( \int_{\mathbb{R}} |K(x,y) f(y) | \, dy \right)^p \, dx \leq \int_{\mathbb{R}} B^{p/q} \int_{\mathbb{R}} |K(x,y)| \cdot |f(y)|^{p} \, dy \, dx$$

$$= B^{p/q} \int_{\mathbb{R}} |f(y)|^{p} \left( \int_{\mathbb{R}} |K(x,y)| \, dx \right) \, dy \leq B^{p/q} \int_{\mathbb{R}} |f(y)|^{p} \cdot B \, dy = B^{\frac{q}{p}+1} \cdot |f|_{L^p}^p$$
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Since \((B^{\frac{p+1}{p}})^{1/p} = B^{\frac{1}{p} + \frac{1}{q}} = B^1 = B\), we have the assertion. 

\\[12.4\] (A simple case of Young’s inequality) In the previous example, let \(K(x, y) = k(x - y)\) for \(k \in \mathcal{L}^p(\mathbb{R})\), so that \(Tf(x) = (k * f)(x)\). Show that \(|Tf|_{L^p} \leq |k|_{L^1} \cdot |f|_{L^p}\).

Discussion: A simple case of the previous.

\\[12.5\] Solve \(-u'' + u = \delta\) on \(\mathbb{R}\). (Hint: use Fourier transform. Knowing how to evaluate standard/iconic integrals by residues would be convenient, but/and the relevant integral was done in an earlier example-discussion.)

Discussion: Let’s assume that we are asking for a solution \(u\) that is at worst a tempered distribution. Thus, we can take Fourier transform, obtaining

\[ (4\pi^2\xi^2 + 1)\hat{u} = \hat{\delta} = 1 \]

Obviously we want to divide by \(4\pi^2\xi^2 + 1\). Unlike some other examples, where division was not quite legitimate, here, we can achieve the effect by multiplication by the smooth, bounded function \(1/(4\pi^2\xi^2 + 1)\), since \(4\pi^2\xi^2 + 1\) does not vanish on \(\mathbb{R}\). Thus,

\[ \hat{u} = \frac{1}{4\pi^2\xi^2 + 1} \]

Since the right-hand side is luckily in \(\mathcal{L}^1(\mathbb{R})\), we can compute its image under Fourier inversion by the literal integral, its inverse Fourier transform will be a continuous function (by Riemann-Lebesgue), so has meaningful pointwise values:

\[ u(x) = \int_{\mathbb{R}} \frac{e^{2\pi i \xi x}}{4\pi^2\xi^2 + 1} d\xi \]

The integral can be evaluated by residues: depending on the sign of \(x\), we use an auxiliary arc in the upper (for \(x > 0\)) or lower (for \(x < 0\)) half-plane, so that \(\xi \rightarrow e^{2\pi i \xi} x\) is bounded in the corresponding half-plane. Thus, we pick up either \(2\pi i\) times the residue at \(\xi = 1/2\pi i\), or the negative (because the orientation is negative) of the residue at \(\xi = -1/2\pi i\). That is, respectively,

\[ 2\pi i \cdot \frac{e^{2\pi i (1/2\pi i) x}}{4\pi^2 \cdot (\frac{1}{2\pi i} - \frac{1}{2\pi i})} = -e^{-x} = -e^{-|x|} \quad \text{(for} x \geq 0\text{)} \]

and

\[ -2\pi i \cdot \frac{e^{2\pi i (-1/2\pi i) x}}{4\pi^2 \cdot (\frac{-1}{2\pi i} - \frac{1}{2\pi i})} = -e^x = -e^{-|x|} \quad \text{(for} x \leq 0\text{)} \]

\\[12.6\] Show that \(u'' = \delta\) has no solution on the circle \(T\). (Hint: Use Fourier series.) Show that \(u'' = \delta - 1\) does have a solution. (And reflect on the Fredholm alternative?)

Discussion: In Fourier series converging in \(H^{-\frac{1}{2} - \varepsilon}(T)\) for all \(\varepsilon > 0\), \(\delta = \sum_{n \in \mathbb{Z}} 1 \cdot \psi_n\), where \(\psi_n(x) = e^{2\pi i nx}\). A function \(u\) in the relatively large-yet-tractable space \(H^{-\infty}(T)\) has a Fourier expansion \(u = \sum_n \tilde{u}(n) \cdot \psi_n\). Application of the (extended-sense) second derivative operator can be done termwise (by design), and annihilates the \(n = 0\) term. That is, no \(u''\) can have \(0^{th}\) Fourier coefficient 1, as does \(\delta\), so that equation is not solvable.

In contrast, \(\delta - 1\) has exactly lost that difficult Fourier component, and, in terms of Fourier series, \(u'' = \delta - 1\) is

\[ \sum_{n \in \mathbb{Z}} (2\pi i n)^2 \cdot \tilde{u}(n) \cdot \psi_n = \sum_{n \neq 0} 1 \cdot \psi_n \]

has the solution by division

\[ u = \sum_{n \neq 0} \frac{1}{(2\pi i n)^2} \psi_n \]

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On the circle $T$, show that $u'' = f$ has a unique solution for all $f \in L^2(T)$ orthogonal to the constant function $1$. (And reflect on the Fredholm alternative?)

**Discussion:** The orthogonality to $1$ means that the $0^{th}$ Fourier coefficient of $f$ is $0$. Thus, on the Fourier series side, for any $u \in H^{-\infty}(T)$, $u'' = f$ is

$$
\sum_{n \in \mathbb{Z}} (2\pi in)^2 \cdot \hat{u}(n) \cdot \psi_n = \sum_{n \neq 0} \hat{f}(n) \cdot \psi_n
$$
gives

$$
u = \sum_{n \neq 0} \frac{\hat{f}(n)}{(2\pi in)^2} \cdot \psi_n
$$
and there is no other solution in $H^{-\infty}(T)$. ///