Examples discussion 13

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[This document is http://www.math.umn.edu/~garrett/m/real/examples_discussion_13.pdf]

[13.1] On $\mathbb{R}^2$, compute the Fourier transform of $(x \pm iy)^n e^{-\pi(x^2+y^2)}$ for $n = 0, 1, 2, \ldots$ (Hint: Re-express things, including Fourier transform, in terms of $z = x + iy$ and $\overline{z} = x - iy$, $w = u + iv$, and $\overline{w} = u - iv$.)

**Discussion:** Using $z$ and $w$, the functions are $z^n e^{-\pi z \overline{z}}$ and $\overline{z}^n e^{-\pi \overline{z} z}$, and Fourier transform is

$$\int_{\mathbb{R}^2} e^{-\pi i(z\overline{\zeta} + \zeta \overline{w})} z^n e^{-\pi z \overline{z}} \, dx \, dy = \int_{\mathbb{R}^2} e^{-\pi i(z\overline{\zeta} + \zeta \overline{w})} \left(\frac{1}{-\pi}\right)^n \left(\frac{\partial}{\partial z}\right)^n e^{-\pi z \overline{z}} \, dx \, dy$$

Imagining that we can integrate by parts, this is

$$(-1)^n \frac{1}{(-\pi)^n} \int_{\mathbb{R}^2} \left(\frac{\partial}{\partial z}\right)^n e^{-\pi i(z\overline{\zeta} + \zeta \overline{w})} e^{-\pi z \overline{z}} \, dx \, dy = \frac{1}{\pi^n} \int_{\mathbb{R}^2} (-\pi i w)^n e^{-\pi i(z\overline{\zeta} + \zeta \overline{w})} e^{-\pi z \overline{z}} \, dx \, dy$$

$$= (-i)^n w^n \int_{\mathbb{R}^2} e^{-\pi i(z\overline{\zeta} + \zeta \overline{w})} e^{-\pi z \overline{z}} \, dx \, dy = i^{-n} w^n e^{-\pi |w|^2}$$

since we know the Fourier transform of a Gaussian. A similar computation with roles of $z, \overline{z}$ reversed accomplishes the other computation. That is, $(x \pm iy)^n e^{-\pi(x^2+y^2)}$ is an eigenfunction for Fourier transform, with eigenvalue $i^{-|n|}$. ///

[13.2] Let $S, T$ be two compact, self-adjoint operators on a Hilbert space, and $ST = TS$. Show that there is an orthonormal basis for $V$ consisting of simultaneous eigenfunctions for $S, T$.

**Discussion:** The Hilbert space $V$ is the closure of the orthogonal direct sum of eigenspaces $V_\lambda$ for $T$. For $\lambda \neq 0$, $V_\lambda$ is finite-dimensional, so is necessarily closed, and $V_0$ is the orthogonal complement of the sum of all other eigenspaces, so is closed. Since $ST = TS$, we find that $S$ stabilizes each $V_\lambda$:

$$T(Sv) = (TS)v = (ST)v = S(Tv) = S(\lambda v) = \lambda \cdot Sv \quad \text{(for all } v \in V_\lambda)$$

[0.1] **Claim:** The restriction of a compact operator to a closed subspace $W \subset V$ stabilized by it is still compact.

**Proof:** With $B'$ the closed unit ball of $W$ and $B$ the closed unit ball of $V$, $TB' \subset TB$. Using the total-boundedness criterion for precompactness, given $\varepsilon > 0$, $TB$ is covered by finitely-many $\varepsilon$-balls $v_i + B_\varepsilon$. Among the intersections $W \cap (v_i + B_\varepsilon)$, the non-empty ones are open balls of radius at most $\varepsilon$. Thus, $TB'$ is a precompact set, and $T|_W$ is a compact operator.

Thus, $S$ is a compact operator on each $V_\lambda$, so every $V_\lambda$ has an orthonormal basis of $S$-eigenvectors. These are also $\lambda$-eigenvalues for $T$, so they are simultaneous eigenvectors. ///

[13.3] Show that $\varphi \to \int_{\mathbb{R}} e^{x^2} \varphi(x) \, dx$ is a distribution.

**Discussion:** From the characterization of the topology on $D$, a compatible family of continuous linear functionals $\lambda_K$ on the Fréchet spaces

$$D_K = \{ f \in D : \text{spt } f \subset K \}$$

uniquely determines a continuous linear function on $D$. The **compatibility** condition is that, for $K \subset K'$, with inclusion map $i_K^{K'} : D_K' \to D_K$, we have $\lambda_K' \circ i_K^{K'} = \lambda_K$. For $\varphi \in D_K$,

$$\left| \int_{\mathbb{R}} e^{x^2} \varphi(x) \, dx \right| \leq \left( \sup_{x \in K} e^{x^2} \right) \cdot \int_{\mathbb{R}} |\varphi(x)| \, dx \leq \left( \sup_{x \in K} e^{x^2} \right) \cdot \text{meas } (K) \cdot \sup_{x \in K} |\varphi(x)|$$
Since \( \sup_K |\varphi(x)| \) is one of the seminorms defining the Fréchet space topology on \( D_K \), this proves the continuity of that functional on \( D_K \). \( \text{Compatibility} \) is clear, although the constants appearing in the corresponding estimate depend on the compacts. \///

[13.4] Show that \( \varphi \rightarrow \sum_{0 \leq n \in \mathbb{Z}} \varphi^{(n)}(n) \) is a distribution.

**Discussion:** As in the previous example, and since \( \text{compatibility} \) of the functionals on \( D_K \) is again clear, it suffices to prove that the indicated function is continuous on every \( D_K \). For \( \varphi \in D_K \),

\[
\sum_{0 \leq n \in \mathbb{Z}} \varphi^{(n)}(n) = \sum_{0 \leq n \in \mathbb{Z} \cap K} \varphi^{(n)}(n)
\]

which is a **finite** linear combination of translates of derivatives of \( \delta \). A finite linear combination of translates of derivatives of a distribution is again a distribution, and restriction to \( D_K \) gives a continuous linear functional on \( D_K \), since also \( D_K \rightarrow \mathcal{D} \) is continuous. Thus, the original functional is continuous on \( \mathcal{D} \), and thus a distribution. \///

[13.5] (Without invoking classification of distributions supported at a point) show that \( \varphi \rightarrow \sum_{0 \leq n \in \mathbb{Z}} \varphi^{(n)}(0) \) is not a distribution.

**Discussion:** Since \( D_K \rightarrow \mathcal{D} \) is continuous, if the indicated map gave a distribution, it would give a continuous linear functional on every \( D_K \). We will show that this apparent functional does not give a continuous linear functional on any \( D_K \) where \( K \ni 0 \). It suffices to exhibit a test function \( \varphi \) such that the finite partial sums \( u_N(\varphi) = \sum_{n \leq N} \varphi^{(n)}(0) \) do not converge.

The function \( f(x) = \sum_{j \geq 0} \frac{2^j x^j}{j!} \) is smooth on \( R \). Let \( \varphi_o \) be a test function that is identically 1 on a neighborhood of 0. Then \( \varphi(x) = \varphi_o(x) \cdot f(x) \) is a test function whose derivatives evaluated at 0 are those of \( f \), namely, \( 2^j \). Then \( u_N(\varphi) = 1 + 2 + 4 + 8 + \ldots + 2^N \). The sequence of these goes to \( +\infty \). \///

[0.2] Remark: In fact, \( \varphi \rightarrow \sum_{n \geq 0} \frac{\varphi^{(n)}(0)}{n!} \) is not a distribution, and, further, for no sequence \( c_1, c_2, \ldots \) of non-zero numbers (no matter how rapidly decreasing) is \( \varphi \rightarrow \sum_{n \geq 0} c_n \varphi^{(n)}(0) \) a distribution. We certainly know this from the classification, on one hand. On another hand, a theorem of E. Borel asserts that, for any sequence \( b_n \) of complex numbers, there is a smooth function whose Taylor coefficients at 0 are the given sequence of numbers. In particular, this applies to \( b_n = e^{c^n} \) or any other sequence, no matter how rapidly increasing.

[13.6] Let \( T \) be a continuous/bounded self-adjoint operator on a Hilbert space \( V \), with spectrum consisting of just two points \( \lambda \neq \mu \). Show that the isomorphism \( C^0(\{\lambda, \mu\}) \approx \mathbb{R}[T] \) implies that \( V \) is the direct sum of \( \lambda \) and \( \mu \) eigenspaces.

**Discussion:** Let \( f, g \) be continuous real-valued functions such that \( f(\lambda) = 1, f(\mu) = 0, g(\lambda) = 0, \) and \( g(\mu) = 1 \). On the spectrum of \( T \), \( (x - \lambda)f(x) = 0 \), so \( (T - \lambda)f(T) = 0 \), so \( T \) acts by \( \lambda \) on the image \( f(T)V \) of \( f(T) \). Similarly, \( T \) acts by \( \mu \) on the image \( g(T)V \) of \( g(T) \). On the spectrum of \( T \), \( f^2(x) = f(x) \), since both are 1 on \( \lambda \) and 0 on \( \mu \). Similarly, \( g^2(x) = g(x) \) on the spectrum. Since \( f(T) \) and \( g(T) \) are in the operator-norm closure of \( \mathbb{R}[T] \), they are self-adjoint. Self-adjoint idempotent operators are orthogonal projectors, so \( f(T) \) is a (non-zero, because \( f \) restricted to the spectrum is not identically 0) orthogonal projector to some (non-zero) subspace of the \( \lambda \)-eigenspace, and similarly for \( g(T) \).

Since \( f + g = 1 \) on the spectrum of \( T \), \( f(T) \) must map to the whole \( \lambda \)-eigenspace, and similarly for \( g \), and the (orthogonal) sum of these eigenspaces is the whole Hilbert space. \///

[0.3] Remark: One should prove that an operator-norm limit of self-adjoint functions is self-adjoint.