Review examples 01

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If you want feedback on your write-ups on any of these examples, please get your write-ups by Monday, 19 Sept, 2016.

[01.1] Prove (or review the proof) that a uniform pointwise limit of continuous, real-valued functions on $[a,b]$ is continuous.

[01.2] Prove (or review the proof) of the Fundamental Theorem of Calculus: for a continuous function $f$ on $[a,b]$, the function $F(x) = \int_a^x f(t) \, dt$ is continuously differentiable, and has derivative $f$. (Use Riemann’s integral.)

[01.3] Prove (or review the proof) that for a sequence of real-valued functions $f_n$ on $[0,1]$ approaching $f$ uniformly pointwise, $\lim_n \int_0^1 f_n(x) \, dx = \int_0^1 \lim_n f_n(x) \, dx$. (Use Riemann’s integral.)

[01.4] Show that every open subset of $\mathbb{R}$ is a countable union of open intervals.

[01.5] Define an (outer) measure $\mu(E)$ of subsets $E$ of $\mathbb{R}$ given by

$$
\mu(E) = \inf \left\{ \sum_{n=1}^{\infty} |b_n - a_n| : E \subset \bigcup_{n=1}^{\infty} (a_n, b_n) \right\}
$$

Show that $\mu(\mathbb{Q}) = 0$. Show that $\mu(M) = 0$, where $M$ is Cantor’s middle-thirds set.

[01.6] Give an example of a sequence of continuous real-valued functions $\{f_n\}$ on $[0,1]$ whose pointwise limit $f(x) = \lim_n f_n(x)$ is $1/q$ on rational numbers $p/q$ in lowest terms, and is 0 for $x$ irrational.

The following examples are essentially irrelevant to us, but have some entertainment value, and would once have been considered significant. But by now, if anything, such examples and counter-examples tend to illustrate the futility of otherwise-appealing enterprises.

[01.7] (*) Given an enumeration $r_1, r_2, \ldots$ of rational numbers in $[0,1]$, and given a sequence $y_1, y_2, \ldots \to 0$ of real numbers going to 0, construct a sequence of continuous real-valued functions $\{f_n\}$ on $[0,1]$ whose pointwise limit $f(x) = \lim_n f_n(x)$ is $y_n$ on $r_n$, and is 0 for $x$ irrational.

[01.8] (*) Use this to construct a sequence $\{\{f_{mn} : n = 1, 2, \ldots \} : m = 1, 2, \ldots \}$ of sequences such that $\lim_m (\lim_n f_{mn}(x)) = 1$ for rational $x$, and 0 for irrational $x$.

[01.9] (**) Show that for any sequence $\{f_n\}$ of real-valued functions on $[0,1]$ with $0 \leq f_n(x) \leq 1$ for all $x, n$, and with $f_n(x) \to 1$ for rational $x$, there are uncountably-many $y \in [0,1]$ with $\limsup f_n(y) = 1$. 

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