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Review examples discussion 00

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[This document is http://www.math.umn.edu/~garrett/m/real/notes_2017-18/real-disc-00.pdf]

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[00.1] (There is not much hope in making sense of the outcome of an uncountable number of non-zero operations:) Let \( \Omega \) be an uncountable collection of positive real numbers. Letting \( F \) range over all finite subsets of \( \Omega \), show that \( \sup_{F} \sum_{\alpha \in F} \alpha = + \infty \).

**Discussion:** Let \( \Omega_{n} = \{ \omega \in \Omega : \omega > 1 \} \), and for \( n = 2, 3, \ldots \), let \( \Omega_{n} = \{ \omega \in \Omega : \frac{1}{n} < \omega \leq \frac{1}{n-1} \} \). There are countably many such sets, so in (at least) one of them \( \Omega_{n} \) subsets of \( \Omega \), show that \( \sup \cdot \sum_{\alpha \in F} \alpha = + \infty \) because \( \Omega_{n} \) is infinite.

\[
\sup_{F} \sum_{\alpha \in F} \geq \sup_{F \subseteq \Omega_{n}^{\alpha}} \sum_{\alpha \in F} \geq \sup_{F \subseteq \Omega_{n}^{\alpha}} \# F \cdot \frac{1}{n_{\alpha}} \cdot \frac{1}{n_{\omega}} \cdot \frac{1}{F \subseteq \Omega_{n}^{\alpha}} \cdot \# F = + \infty
\]

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[00.2] Prove (or review the proof) that a continuous real-valued function \( f \) on a finite closed interval \([a, b] \subset \mathbb{R} \) is uniformly continuous: for all \( \varepsilon > 0 \) there is \( \delta > 0 \) such that, for all \( x, y \in [a, b] \), \( |x - y| < \delta \) implies \( |f(x) - f(y)| < \varepsilon \).

**Discussion:** Given \( \varepsilon > 0 \) and \( x \in [a, b] \), take \( \delta_{x} > 0 \) such that \( |x' - x| < 2\delta_{x} \) implies \( |f(x') - f(x)| < \varepsilon/2 \). The open intervals \( (x - \delta_{x}, x + \delta_{x}) \) cover the compact set \([a, b] \), so there is a finite subcover \( \{(x_{j} - \delta_{x_{j}}, x_{j} + \delta_{x_{j}}) : j = 1, \ldots, N \} \). The minimum \( \delta = \min_{j=1,\ldots,N} \delta_{j} \) is positive (see above). For given \( x \in [a, b] \), \( x \in (x_{j} - \delta_{x_{j}}, x_{j} + \delta_{x_{j}}) \) for some \( j \).

For \( x' \) such that \( |x' - x| < \delta \), we have \( |x' - x_{j}| \leq |x' - x| + |x - x_{j}| \leq \delta + \delta_{j} \leq 2\delta_{j} \), so \( |f(x') - f(x_{j})| < \varepsilon/2 \), and

\[
|f(x') - f(x)| \leq |f(x') - f(x_{j})| + |f(x_{j}) - f(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon
\]

which is the uniform continuity.

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[00.3] Prove (or review the proof) that a uniform pointwise limit of continuous, real-valued functions on \([a, b] \) is continuous.

**Discussion:** This is the archetype of a three-\( \varepsilon \) argument. Let the sequence by \( \{ f_{n} \} \), and the pointwise limit \( f(x) = \lim_{n} f_{n}(x) \). Given \( \varepsilon > 0 \), by the uniform pointwise approach to the limit, take \( n_{\alpha} \) large enough so that for all \( m, n \geq n_{\alpha} \), for all \( x \in [a, b] \), \( |f_{m}(x) - f_{n}(x)| < \varepsilon \). Then \( |f(x) - f_{n}(x)| \leq \varepsilon \) for all \( x \in [a, b] \), for all \( n \geq n_{\alpha} \). By the uniform continuity of \( f_{n} \) on \([a, b] \), let \( \delta > 0 \) so that \( |f_{n}(x) - f_{n}(y)| < \varepsilon \) for all \( |x - y| < \delta \).

Then

\[
|f(x) - f(y)| \leq |f(x) - f_{n}(x)| + |f_{n}(x) - f_{n}(y)| + |f_{n}(y) - f(y)| < \varepsilon + \varepsilon + \varepsilon
\]

as desired.

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**Note:** In the latter situation, there is no compulsion to go back and replace \( \varepsilon \) by \( \varepsilon/3 \), since it is obviously possible to do so.

[00.4] Prove (or review the proof) of the Fundamental Theorem of Calculus: for a continuous function \( f \) on \([a, b] \), the function \( F(x) = \int_{a}^{x} f(t) \, dt \) is continuously differentiable, and has derivative \( f \). (Use Riemann’s integral.)
**Discussion:** We use the finite additivity property

\[
\int_a^c f(x) \, dx = \int_a^v f(x) \, dx + \int_v^c f(x) \, dx \quad \text{(for all } v < c \text{ between } a \text{ and } b)\]

Thus,

\[
\frac{F(x + \delta) - F(x)}{\delta} - f(x) = \frac{\int_x^{x+\delta} f(t) \, dt}{\delta} - f(x)
\]

By continuity of \( f \), given \( \varepsilon > 0 \), take \( \delta_\varepsilon > 0 \) sufficiently small so that

\[
\sup_{y:x \leq y \leq x + \delta_\varepsilon} |f(y) - f(x)| < \varepsilon
\]

Then

\[
\frac{\int_x^{x+\delta_\varepsilon} f(t) \, dt}{\delta} - f(x) < \frac{(f(x) + \varepsilon) \cdot \delta}{\delta} - f(x) = \varepsilon
\]

and, similarly,

\[
\frac{\int_x^{x+\delta_\varepsilon} f(t) \, dt}{\delta} - f(x) > \frac{(f(x) - \varepsilon) \cdot \delta}{\delta} - f(x) = -\varepsilon
\]

Thus, given \( \varepsilon > 0 \), there is \( \delta_\varepsilon > 0 \) such that for every \( 0 < \delta \leq \delta_\varepsilon \)

\[
\left| \frac{F(x + \delta) - F(x)}{\delta} - f(x) \right| < \varepsilon
\]

(Finding \( \delta_\varepsilon < 0 \) for the same inequality is similar.)

[The following is a discussion of the question I meant to ask!!!]

**[00.5]** Prove (or review the proof) that for a sequence of real-valued functions \( f_n \) on \([0,1]\) approaching \( f \) uniformly pointwise, \( \lim_{n \to \infty} \int_0^1 f_n(x) \, dx = \int_0^1 f(x) \, dx \). (Use Riemann’s integral.)

**Discussion:** Given \( \varepsilon > 0 \), let \( n_\varepsilon \) be large enough so that for all \( n \geq n_\varepsilon \), for all \( x \in [a,b] \), \( |f_n(x) - f(x)| < \varepsilon \). Using linearity of integrals,

\[
\int_a^b f(x) \, dx = \int_a^b f(x) - f_{n_\varepsilon}(x) \, dx + \int_a^b f_{n_\varepsilon}(x) \, dx
\]

Upper and lower bounds are obtained from any upper and lower Riemann sums, for any partition \( a = x_1 < \ldots < x_n = b \) of the interval:

\[
\int_a^b f(x) - f_{n_\varepsilon}(x) \, dx < \sum_{j=1}^n (x_{j+1} - x_j) \cdot \varepsilon = (b-a) \cdot \varepsilon
\]

and similarly for a lower bound.

[///]

**[00.6]** Show that every open subset of \( \mathbb{R} \) is a countable union of open intervals.

**Discussion:** Let \( S \) be the set. For \( s \in S \), since \( S \) is open, there is \( 0 < \delta_s \in \mathbb{Q} \) such that \( (s - 2\delta_s, s + 2\delta_s) \subset S \). By density of \( \mathbb{Q} \) in \( \mathbb{R} \) there is \( q_s \) in the smaller interval \( (s - \delta_s, s + \delta_s) \). Certainly \( s \in (q_s - \delta_s, q_s + \delta_s) \), and \( (q_s - \delta_s, q_s + \delta_s) \subset S \), because for \( |t - q_s| < \delta_s \)

\[
|s - t| \leq |s - q_s| + |q_s - t| < \delta + \delta
\]
The collection of all pairs \((q, \delta) \in \mathbb{Q} \times \mathbb{Q}\) of rationals \(q, \delta\) is countable, so the subset of (distinct) pairs occurring as \(q_s, \delta_s\) for \(s \in S\) is countable. (Apparently many of the pairs \((q, \delta)\) appear as \((q_s, \delta_s)\) for many different \(s \in S\).)

[0.7] Define Lebesgue (outer) measure \(\mu(E)\) of subsets \(E\) of \(\mathbb{R}\) given by

\[
\mu(E) = \inf\{\sum_{n=1}^{\infty} |b_n - a_n| : E \subset \bigcup_{n=1}^{\infty}(a_n, b_n)\}
\]

Show that \(\mu(\mathbb{Q}) = 0\). Show that \(\mu(M) = 0\), where \(M\) is Cantor’s middle-thirds set.

Discussion: Enumerate the rationals as \(r_1, r_2, \ldots\). Given \(\varepsilon > 0\), let \(U_{n, \varepsilon}\) be the interval \((r_n - \frac{\varepsilon}{2^n}, r_n + \frac{\varepsilon}{2^n})\). The union of these intervals contains \(\mathbb{Q}\), and the sum of lengths is \(\varepsilon \cdot (\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots) = \varepsilon\).

The Cantor middle-thirds set can be described in terms of base-three expansions, as follows. All real numbers \(r\) in \([0, 1]\) have (ternary) expansion \(r = \sum_{n=1}^{\infty} \frac{a_n}{3^n}\) with all coefficients \(a_n\) in the set \(\{0, 1, 2\}\). The expansion is unambiguous except for the possibility of coefficients all 2 beyond a certain point, which we exclude by using

\[
\frac{2}{3^n} + \frac{2}{3^{n+1}} + \frac{2}{3^{n+2}} + \ldots = 2 \cdot \frac{3^{-n}}{1 - \frac{1}{3}} = 2 \cdot \frac{3^{1-n}}{3-1} = 3^{1-n}
\]

Then the middle-thirds set \(C\) is the set of reals \(r = \sum_{n=1}^{\infty} \frac{a_n}{3^n}\) with all coefficients \(a_n\) in the set \(\{0, 2\}\) (with the convention excluding endlessly repeating 2’s).

Alternatively, the middle-thirds set \(C\) is formed as a nested intersection, as follows. Let \(C_1\) be \([0, 1]\) with the middle third \((\frac{1}{3}, \frac{2}{3})\) removed. Let \(C_2\) be \(C_1\) with the middle thirds \((\frac{1}{9}, \frac{2}{9})\) and \((\frac{7}{9}, \frac{8}{9})\) removed, and so on. At each step, the sum of lengths of the remaining intervals is multiplied by \((1 - \frac{1}{3}) = \frac{2}{3}\), and the number of intervals is multiplied by 2. After \(n\) middle-third removals, the result \(C_n\) is a union of \(2^n\) intervals each of length \(3^{-n}\). The Cantor middle-thirds set is \(C = \bigcap_n C_n\).

Given \(\varepsilon > 0\), choose \(n\) large enough so that \(2^n/3^n < \varepsilon/2\). Cover each of the \(2^n\) intervals of length \(3^{-n}\) making up \(C_n\) by an open interval of length \(2 \cdot 3^{-n}\). The sum of the lengths of these \(2^n\) open intervals is

\[
2^n \cdot (2 \cdot 3^{-n}) = 2 \cdot (2/3)^n < 2 \cdot \frac{\varepsilon}{2} = \varepsilon
\]

This exhibits an open cover of \(C_n\) with sum of lengths less than \(\varepsilon\). Since \(C \subset C_n\), this gives such a cover of \(C\) itself, as desired.