Examples discussion 02

[This document is http://www.math.umn.edu/~garrett/m/real/examples_2017-18/real-disc-02.pdf]

[02.1] Show that $\ell^2$ is complete as a metric space.

Discussion: We can do this directly, although it is also a special case of the general fact that $L^2(X,\mu)$ is complete. Indeed, the argument will be a somewhat simpler version of the more general proof.

Let $f_1, f_2, \ldots$ be a Cauchy sequence in $\ell^2$. Let $f(n)$ be the $n^{th}$ component of $f \in \ell^2$, for $n = 1, 2, \ldots$. For any $f \in \ell^2$, certainly $|f(n)| \leq |f|_{\ell^2}$, so for each $n$ the scalar sequence $f_1(n), f_2(n), f_3(n), \ldots$ must be Cauchy, thus has a limit $f(n)$. We claim that $f = (f(1), f(2), f(3), \ldots)$ is in $\ell^2$, and is the $\ell^2$ limit of the $f_i$.

Given $\varepsilon > 0$, there is $N$ sufficiently large so that $|f_i - f_j|_{\ell^2} < \varepsilon$ for all $i, j \geq N$. By a discrete version of Fatou’s lemma, for $i \geq N$,

$$\sum_{n} |f(n) - f_i(n)|^2 = \sum_{n} \lim_j |f_j(n) - f_i(n)|^2 = \sum_{n} \liminf_j |f_j(n) - f_i(n)|^2 \leq \liminf_j \sum_{n} |f_j(n) - f_i(n)|^2 \leq \liminf_j |f_j - f_i|_{\ell^2} \leq \liminf_j \varepsilon^2 = \varepsilon^2$$

Thus, $f - f_i \in \ell^2$, so $f = (f - f_i) + f_i \in \ell^2$. Then the previous computation shows that for given $\varepsilon$ for $i \geq N$ we have $|f - f_i| \leq \varepsilon$. Thus, $f_i \to f$ in $\ell^2$.

Discrete version of Fatou’s Lemma: We claim that for $[0, +\infty]$-valued functions $f_j$ on $\{1, 2, 3, \ldots\}$,

$$\sum_{n=1}^{\infty} \liminf_{j} f_j(n) \leq \liminf_{j} \sum_{n=1}^{\infty} f_j(n)$$

Proof: Letting $g_j(n) = \inf_{j \geq n} f_j(n)$, certainly $g_j(n) \leq f_j(n)$ for all $n$, and $\sum_{n} g_j(n) \leq \sum_{n} f_j(n)$. Also, $g_1(n) \leq g_2(n) \leq \ldots$ for all $n$, and $\lim_{j} g_j(n) = \lim_{j} f_j(n)$. A discrete form of the Monotone Convergence Theorem, proven just below, is

$$\lim_{j} \sum_{n} g_j(n) = \sum_{n} \lim_{j} g_j(n)$$

Thus,

$$\sum_{n} \liminf_{j} f_j(n) = \sum_{n} \lim_{j} g_j(n) = \lim_{j} \sum_{n} g_j(n) = \liminf_{j} \sum_{n} g_j(n) \leq \liminf_{j} \sum_{n} f_j(n)$$

as claimed.

Similarly, we have

Discrete version of Lebesgue’s Monotone Convergence Theorem: For $[0, +\infty]$-valued functions $f_j$ on $\{1, 2, 3, \ldots\}$, with $f_1(n) \leq f_2(n) \leq \ldots$ for all $n$,

$$\lim_{j} \sum_{n=1}^{\infty} f_j(n) = \sum_{n=1}^{\infty} \lim_{j} f_j(n) \quad \text{(allowing value } +\infty)$$

Proof: Each non-decreasing sequence $f_1(n) \leq f_2(n) \leq \ldots$ has a limit $f(n) \in [0, +\infty]$. Similarly, since $\sum_{n} f_j(n) \leq \sum_{n} f_{j+1}(n)$, the non-decreasing sequence of these sums has a limit $S = \lim_{j} \sum_{n} f_j(n)$. Since $f_j(n) \leq f(n)$, certainly $\sum_{n} f_j(n) \leq \sum_{n} f(n)$, and $S \leq \sum_{n} f(n)$. 


Fix $N$, and put $g(n) = f(n)$ for $n \leq N$ and $g(n) = 0$ for $n > N$. For $\varepsilon > 0$, let

$$E_j = \{ n : \sum_n f_j(n) \geq (1 - \varepsilon) \cdot \sum_n g(n) \} \quad \text{(for } j = 1, 2, \ldots \}$$

Certainly $E_1 \subset E_2 \subset \ldots$, since $f_{j+1}(n) \geq f_j(n)$ for all $n$. We claim that $\bigcup E_j = \{ 1, 2, \ldots \}$: for $f(n) > 0,

$$\lim_j f_j(n) = f(n) > (1 - \varepsilon) \cdot f(n) \geq (1 - \varepsilon) \cdot g(n) \quad \text{(for all } n)$$

and for $f(n) = 0$, also $g(n) = 0$, and

$$f_1(n) \geq 0 \geq (1 - \varepsilon) g(n)$$

Then

$$\sum_n f_j(n) \geq \sum_{n \in E_j} f_j(n) \geq (1 - \varepsilon) \cdot \sum_{n \in E_j} g(n)$$

The set of $n$ for which $g(n)$ is non-zero is finite, so there is $j_0$ such that for $j \geq j_0$

$$\sum_{n \in E_j} g(n) = \sum_n g(n) \quad \text{(for all } j \geq j_0)$$

That is, $\lim_j \sum_n f_j(n) \geq (1 - \varepsilon) \sum_n g(n)$. Then

$$S = \lim_j \sum_n f_j(n) \geq (1 - \varepsilon) \cdot \lim_j \sum_{n \in E_j} g(n) = (1 - \varepsilon) \cdot \sum_n g(n)$$

This holds for every $\varepsilon > 0$, so $S \geq \sum_n g(n) = \sum_{n \leq N} f(n)$. This holds for every $N$, so $S \geq \sum_n f(n)$. ///

[02.2] Show that the characteristic function $\chi_E$ of a measurable set $E$ is measurable.

Discussion: For non-empty open $U \subset \mathbb{R}$, $\chi_E^{-1}(U)$ is the measurable set $\phi$ if $U$ does not contain either 0 or 1. If $U \ni 1$ but $U \not\ni 0$, then $\chi_E^{-1}(U) = E$, which is measurable. If $U \ni 0$ but $U \not\ni 1$, then $\chi_E^{-1}(U) = E^c$, the complement of $E$, which is measurable. If $U$ contains both 0 and 1, then $\chi_E^{-1}(U)$ is the whole domain space, which is measurable. ///

[02.3] Show that the product of two $\mathbb{R}$-valued measurable functions on $\mathbb{R}$ is measurable.

Discussion: Let $f, g$ be measurable functions. Let $\Delta : \mathbb{R} \to \mathbb{R}^2$ by $\Delta(x) = (x, x)$, $s : \mathbb{R}^2 \to \mathbb{R}$ by $m(x, y) = x \cdot y$, and $f \oplus g : \mathbb{R}^2 \to \mathbb{R}^2$ by $(f \oplus g)(x, y) = (f(x), g(y))$. Clearly $m \circ (f \oplus g) \circ \Delta = f \cdot g$, and $(f \cdot g)^{-1} = \Delta^{-1} \circ (f \oplus g)^{-1} \circ m^{-1}$.

For open $U \subset \mathbb{R}$, $m^{-1}(U) \subset \mathbb{R}^2$ is open, because $m$ is continuous. Since $\mathbb{R}^2$ is countably based, and in fact has a countable basis consisting of rectangles with rational endpoints, so $m^{-1}(U)$ is a countable unions of rectangles $(a_i, b_i) \times (c_i, d_i)$. Then

$$(f \oplus g)^{-1} \circ m^{-1}(U) = (f \oplus g)^{-1} \left( \bigcup_i (a_i, b_i) \times (c_i, d_i) \right)$$

$$= \bigcup_i (f \oplus g)^{-1}(a_i, b_i) \times (c_i, d_i) = \bigcup_i f^{-1}(a_i, b_i) \times g^{-1}(c_i, d_i)$$

The sets $f^{-1}(a_i, b_i) \subset \mathbb{R}$ and $g^{-1}(c_i, d_i) \subset \mathbb{R}$ are Borel sets, so their product is a Borel set in $\mathbb{R}^2$. Then

$$\Delta^{-1}(E_1 \times E_2) = E_1 \cap E_2 \quad \text{(for } E_1, E_2 \text{ measurable in } \mathbb{R})$$
is measurable.

[02.4] Use Urysohn’s lemma to prove that \( C^0[a, b] \) is dense in \( L^1[a, b] \).

Discussion: By the Lebesgue definition of integrals, simple functions are dense in \( L^1[a, b] \), so it suffices to show that simple functions can be well approximated by continuous functions. Granting ourselves the (outer and inner) regularity of Lebesgue measure \( \mu \), for measurable \( E \) there are open \( U \) and compact \( K \) such that \( K \subset E \subset U \), and \( \mu(U) - \mu(K) < \varepsilon \). Invoke Urysohn to make a continuous function \( f \) taking values in \([0, 1]\) and \( f|_K = 1 \) and \( f = 0 \) off \( U \). Then

\[
\int_a^b |f - \text{ch}_E| = \int_K |f - \text{ch}_E| + \int_{E-K} |f - \text{ch}_E| + \int_{U-E} |f - \text{ch}_E| \leq \int_K |1 - 1| + \int_{E-K} 1 + \int_{U-E} 1
\]

\[
= \mu(E-K) + \mu(U-E) = \mu(U-K) < \varepsilon
\]
as desired.

[02.5] Comparing \( L^p \) spaces. Let \( 1 \leq p, p' < \infty \). When is \( L^p[a, b] \subset L^{p'}[a, b] \) for finite intervals \([a, b]\) and Lebesgue measure? When is \( L^p(\mathbb{R}) \subset L^{p'}(\mathbb{R}) \)?

Discussion: Take \( p < p' \). We claim that \( L^p[a, b] \supset L^{p'}[a, b] \), with proper containment. The function \( f \) that is \( (x-a)^{-\frac{1}{p'}} \) on \((a, b]\) and 0 off that interval is \( \text{not in } L^{p'} \), but is in \( L^p \). Given \( f \in L^{p'}[a, b] \), let \( E \) be the set of \( x \in [a, b] \) where \( |f(x)| \geq 1 \). Then \( \int_a^b |f|^p < \infty \) if and only if \( \int_E |f|^p < \infty \). On \( E \), \( |f|^p < |f|^{p'} \), so \( \int_E |f|^p < \infty \), and then also \( \int_a^b |f|^p < \infty \), so \( f \in L^p[a, b] \).

We claim that \( L^p(\mathbb{R}) \) and \( L^{p'}(\mathbb{R}) \) are not comparable for \( p \neq p' \). Take \( 1 \leq p < p' \). On one hand, \( 1/(1+|x|)^{1/p'+\varepsilon} \) is in \( L^{p'} \) for all \( \varepsilon > 0 \), but not in \( L^p \) for \( \varepsilon \) small enough so that \( \frac{1}{p'} + \varepsilon < \frac{1}{p} \). On the other hand, the function \( f \) that is \( x^{-\frac{1}{p'}} \) on \((0, 1]\) and 0 off that interval is \( \text{not in } L^p \), but is in \( L^{p'} \).

We claim that for \( 1 \leq p < p' < \infty \), \( \ell^p \subset \ell^{p'} \), with strict containment. Indeed, \( f(n) = 1/n^p \) is not in \( \ell^p \), but is in \( \ell^{p'} \). Let \( E = \{ n \in \{1, 2, \ldots\} : |f(n)| < 1 \} \). Then \( f \in \ell^p \) if and only if the complement of \( E \) is finite, and if \( \sum_{n \in E} |f(n)|^p < \infty \). Certainly \( |f(n)|^p > |f(n)|^{p'} \) for \( n \in E \), and the complement of \( E \) is finite, so \( \sum_{n \in E} |f(n)|^p < \sum_{n \in E} |f(n)|^{p'} \), and \( f \in \ell^{p'} \).

[02.6] For positive real numbers \( w_1, \ldots, w_n \) such that \( \sum_i w_i = 1 \), and for positive real numbers \( a_1, \ldots, a_n \), show that

\[
a_1^{w_1} \cdots a_n^{w_n} \leq w_1 a_1 + \cdots + w_n a_n
\]

Discussion: This is a corollary of Jensen’s inequality, similar to the arithmetic-geometric mean, but with unequal weights. Namely, let \( X = \{1, 2, \ldots, n\} \) with measure \( \mu(i) = w_i \), and function \( f(i) = \log a_i \). Then Jensen’s inequality is

\[
\left( \sum_{i=1}^n w_i \cdot \log a_i \right) = \sum_{i=1}^n w_i \cdot e^{\log a_i}
\]

which simplifies to the assertion.

[02.7] In \( \ell^2 \), show that the point in the closed unit ball closest to a point \( v \) not inside that ball is \( v/|v|_{\ell^2} \).

Discussion: The minimum principle assures that there is a unique closest point \( w \) in the closed unit ball \( B \) to \( v \), because \( B \) is convex, closed, non-empty, and \( v \) is not in \( B \).

Suppose \( w \) is closer than \( v/|v| \). Then

\[
|v|^2 - 2|v| + 1 = |v - \frac{v}{|v|}|^2 > |v-w|^2 = |v|^2 - (v,w) - (w,v) + |w|^2 = |v|^2 - (v,w) - (w,v) + 1
\]
Thus, $2|v| < \langle v, w \rangle + \langle w, v \rangle$

Thus, the sum of the two inner products is positive, and by Cauchy-Schwarz-Bunyakowsky:

$$2|v| < \langle v, w \rangle + \langle w, v \rangle = |\langle v, w \rangle + \langle w, v \rangle| \leq 2|v| \cdot |w|$$

Thus, $1 < |w|$, which is impossible.  

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### [02.8]

For a measurable set $E \subset [0, 2\pi]$, show that

$$\lim_{n \to \infty} \int_E \cos nx \, dx = 0 = \lim_{n \to \infty} \int_E \sin nx \, dx$$

**Discussion:** This is an instance of a Riemann-Lebesgue lemma, namely, that Fourier coefficients of an $L^2$ function on $[0, 2\pi]$ go to 0. Here, the $L^2$ function is the characteristic function of $E$, and we use sines and cosines instead of exponentials.

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### [02.9]

One form of the sawtooth function is $f(x) = x - \pi$ on $[0, 2\pi]$. Compute the Fourier coefficients $\hat{f}(n)$.

**Discussion:** We have the orthonormal basis $e_n(x) = \frac{1}{\sqrt{2\pi}} e^{inx}$ with $n \in \mathbb{Z}$ for the Hilbert space $L^2[0, 2\pi]$. The Fourier coefficients are determined by Fourier’s formula

$$\hat{f}(n) = \int_0^{2\pi} f(x) \frac{e^{-inx}}{\sqrt{2\pi}} \, dx$$

For $n = 0$, this is 0. For $n \neq 0$, integrate by parts, to get

$$\hat{f}(n) = \left[ f(x) \cdot \frac{e^{-inx}}{\sqrt{2\pi} \cdot (-in)} \right]_0^{2\pi} = \left[ \frac{1}{\sqrt{2\pi} \cdot (-in)} \right]_0^{2\pi}$$

$$= \left( \frac{\pi \cdot 1}{\sqrt{2\pi} \cdot (-in)} - \frac{-\pi \cdot 1}{\sqrt{2\pi} \cdot (-in)} \right) = \frac{2\pi}{\sqrt{2\pi} \cdot (-in)} = \frac{\sqrt{2\pi}}{-in}$$

The $L^2$ norm of $f$ is

$$\int_0^{2\pi} (x - \pi)^2 \, dx = \left[ \frac{(x - \pi)^3}{3} \right]_0^{2\pi} = \frac{\pi^3 - (-\pi)^3}{3} = \frac{2\pi^3}{3}$$

Thus, by Parseval,

$$\sum_{n \neq 0} \left| \frac{\sqrt{2\pi}}{-in} \right|^2 = \frac{2\pi^3}{3}$$

This simplifies first to

$$2 \sum_{n \geq 1} \frac{2\pi}{n^2} = \frac{2\pi^3}{3}$$

and then to

$$\sum_{n \geq 1} \frac{1}{n^2} = \frac{\pi^2}{6}$$

That is, Parseval applied to the sawtooth function evaluates $\zeta(2)$.  

///
[02.10] For fixed $y \in [0, 1]$, show that there is no $f_y \in L^2[0, 1]$ so that $\langle g, f_y \rangle = g(y)$ for all $g \in L^2[0, 1]$.

**Discussion:** Part of the issue here is whether $L^2$ functions truly have meaningful pointwise values at all, and we generally imagine that they do not, although such a negative fact may be hard to express formulaically.

Among many approaches, one is to suppose such $f$ exists. Choose an orthonormal basis for $L^2[0, 1]$ consisting of the continuous functions $\psi_n(x) = e^{2\pi i nx}$, and see what the condition $\langle f_y, \psi_n \rangle = \psi_n(y)$ imposes on the alleged $f_y$. Indeed, this condition completely determines the Fourier coefficients of the alleged $f_y$: since $\psi_n \in L^2[0, 1]$, $\langle \psi_n, f_y \rangle = \psi_n(y)$, and then

$$\hat{f}_y(n) = \int_0^1 f_y(x) \overline{\psi_n(x)} \, dx = \langle \psi_n, f_y \rangle = \psi_n(y)$$

so

$$f_y = \sum_{n \in \mathbb{Z}} \overline{\psi_n(y)} \cdot \psi_n \quad \text{(with equality in an $L^2$ sense)}$$

By Parseval,

$$|f_y|_{L^2}^2 = \sum_n |\psi_n(y)|^2 = +\infty$$

since $|\psi_n(y)| = 1$ for all $n$. Thus, there can be no such $f_y$ in $L^2$. ///

**In contrast to the previous example’s outcome:** Let $V$ be the complex vector space of power series $f(z) = \sum_{n \geq 0} c_n z^n$ convergent on the open unit disk $D$ in $\mathbb{C}$, having finite norm

$$|f| = \left( \int_D |f(x + iy)|^2 \, dx \, dy \right)^{\frac{1}{2}}$$

with hermitian inner product

$$\langle f, g \rangle = \int_D f(x + iy) \overline{g(x + iy)} \, dx \, dy$$

It is not hard to show that $\langle z^m, z^n \rangle = 0$ unless $m = n$, in which case it is $\frac{2\pi}{2n+1}$, and that $\psi_n(z) = z^n \cdot \frac{\sqrt{2n+1}}{\sqrt{2\pi}}$ is an orthonormal basis for $V$. The sum $f_w(z) = \sum_{n \geq 0} \psi_n(z) \overline{\psi_n(w)}$ converges absolutely for $z, w \in D$, and

$$\langle g(-), f_w \rangle = g(w) \quad \text{(for $w$ in the disk)}$$

For each fixed $w \in D$, pointwise evaluation $g \to g(w)$ is a continuous linear functional on $V$. 

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5