Examples discussion 06

Paul Garrett  garrett@math.umn.edu  http://www.math.umn.edu/~garrett/

[This document is http://www.math.umn.edu/~garrett/m/m/real/examples_2017-18/real-disc-06.pdf]

[06.1] Given \( f \) in the Schwartz space \( \mathcal{S} \), show that there is \( F \in \mathcal{S} \) with \( F'' = f \) if and only if \( \int_{\mathbb{R}} f = 0 \).

Discussion: On one hand, if \( f = F'' \) for \( F \in \mathcal{S} \), then \( \int_{-\infty}^{x} f(y) \, dy = F(x) \). Since \( \lim_{x \to +\infty} F(x) = 0 \), \( \int_{\mathbb{R}} f = 0 \).

On the other hand, if \( \int_{\mathbb{R}} f = 0 \), let \( F(x) = \int_{-\infty}^{x} f \), and show that \( F \in \mathcal{S} \). Since \( F' = f \) by the fundamental theorem of calculus, the (higher) derivatives of \( F \) are those of \( f \), so all that needs to be shown is that \( F \) itself is of rapid decay. For \( x \to -\infty \),

\[
|F(x)| \leq \int_{-\infty}^{\infty} |f| \leq \int_{-\infty}^{\infty} (1 + y^2)^{-N} \sup_{t \in \mathbb{R}} |(1 + t^2)^N \cdot f(t)| \, dy \leq \sup_{t \in \mathbb{R}} |(1 + t^2)^N \cdot f(t)| \cdot \int_{-\infty}^{\infty} (1 + y^2)^{-N} \, dy
\]

\[
\leq \sup_{t \in \mathbb{R}} |(1 + t^2)^N \cdot f(t)| \cdot \int_{-\infty}^{\infty} \frac{\, dt}{(1 + t^2)^N} \leq \sup_{t \in \mathbb{R}} |(1 + t^2)^N \cdot f(t)| \cdot \frac{1}{|x|^{N-1}}
\]

giving the rapid decay. For \( x \to +\infty \), using the condition \( \int_{\mathbb{R}} f = 0 \),

\[
F(x) = \int_{-\infty}^{x} f = \int_{\mathbb{R}} f - \int_{x}^{\infty} f = 0 - \int_{x}^{\infty} f
\]

so for \( x \to +\infty \) it suffices to similarly estimate

\[
\left| \int_{x}^{\infty} f \right| \leq \int_{x}^{\infty} (1 + y^2)^{-N} \sup_{t \in \mathbb{R}} |(1 + t^2)^N \cdot f(t)| \, dy \leq \sup_{t \in \mathbb{R}} |(1 + t^2)^N \cdot f(t)| \cdot \int_{x}^{\infty} (1 + y^2)^{-N} \, dy
\]

which similarly gives the rapid decay as \( x \to +\infty \).  ///

[06.2] Let \( u(x) = e^x \cdot \sin(e^x) \). Explain in what sense the integral \( \int_{\mathbb{R}} f(x) \, u(x) \, dx \) converges for every \( f \in \mathcal{S} \).

Discussion: The idea is to integrate by parts, noting that \( u = v' \) with \( v(x) = \cos(e^x) \). We must be careful with the boundary terms:

\[
\int_{\mathbb{R}} f(x) \, u(x) \, dx = \int_{\mathbb{R}} f(x) \, v'(x) \, dx = \lim_{M,N \to +\infty} \int_{-M}^{N} f(x) \, v'(x) \, dx
\]

\[
= \lim_{M,N \to +\infty} \left( [f(x) \, v(x)]_{-M}^{N} - \int_{-M}^{N} f'(x) \, v(x) \, dx \right)
\]

Since \( v(x) \) is bounded and \( f' \) is of rapid decay, the limit exists, so the original integral is convergent. Further, the value is correctly determined by integration by parts, namely

\[
- \int_{-\infty}^{\infty} f'(x) \, v(x) \, dx = - \int_{-\infty}^{\infty} f'(x) \, \cos(e^x) \, dx
\]

That is, for \( f \in \mathcal{S} \) and functions such as \( u \) obtained by differentiating bounded smooth functions, integration by parts is completely justifiable via the natural estimates. ///

[06.3] Show that \( \sin(nx) \to 0 \) in the \( \mathcal{D}^\ast \)-topology as \( n \to +\infty \). (Since \( \mathcal{S} \) is strictly larger than \( \mathcal{D} \), this implies that \( \sin(nx) \to 0 \) in the \( \mathcal{D}^\ast \)-topology.)
We must show that, for each $\varphi \in \mathcal{S}$,
\[ \lim_{n \to \infty} \int_{\mathbb{R}} \sin(nx) \varphi(x) \, dx = 0 \]

On one hand, since Schwartz functions are $L^1$, we could invoke Riemann-Lebesgue, since (up to normalizations) the indicated integral is $(\hat{\varphi}(n) - \hat{\varphi}(-n))/2i$.

On another hand, we also know that $\hat{\varphi}$ is again a Schwartz function, so $(\hat{\varphi}(n) - \hat{\varphi}(-n))/2i \to 0$. (Further, if we know that $\mathcal{S}$ is dense in $L^1$, then this gives a slightly different proof of Riemann-Lebesgue.)

---

Let $-\infty < a < b < c < +\infty$, and

\[ f(x) = \begin{cases} 
0 & \text{(for } x < a) \\
A & \text{(for } a < x < b) \\
B & \text{(for } b < x < c) \\
0 & \text{(for } c < x) 
\end{cases} \]

Show that (extended) $\frac{d}{dx} f = A\delta_a + (B - A)\delta_b - B\delta_c$.

---

Show that the principal value functional $u(\varphi) = \text{P.V.} \int_{\mathbb{R}} \frac{\varphi(x)}{x} \, dx$ satisfies $x \cdot u = 1$.

---

Compute the Fourier transform of the sign function
\[ \text{sgn}(x) = \begin{cases} 
1 & \text{(for } x > 0) \\
-1 & \text{(for } x < 0) 
\end{cases} \]
These are direct computations, using the characterizations of multiplication and of derivative by duality. For the first assertion, for things, including Fourier transform, in terms of Fourier transform (up to constants), and it is not quite just an integral. Cannot just divide (pointwise values). Indeed, we have proven that the principal-value integral is the Fourier transform (up to constants), and it is not quite just an integral.

Discussion: From the hint, \( x \cdot (\pi i \sgn) = 1 \). Also, the principal-value functional \( u \) from the previous example satisfies \( x \cdot u = 1 \). Thus,

\[
x \cdot (u - \pi i \sgn) = 0
\]

By another earlier example, this implies that \( u - \pi i \sgn \) is a multiple of \( \delta \). In fact, the multiple is 0, because \( \delta \) is even, while \( u, \sgn \), and thus \( \sgn \), are all odd. \[1\] That is, \( \sgn = \frac{1}{\pi i} u \).

\[0.1\] Remark: In particular, it is not quite that \( \sgn(\xi) = 1/\pi i \xi \). Indeed, \( 1/\xi \) is not locally integrable, so does not directly describe a distribution. This example shows that, yes, \( \xi \cdot \sgn = 1/\pi i \), but apparently we cannot just divide (pointwise values). Indeed, we have proven that the principal-value integral is the Fourier transform (up to constants), and it is not quite just an integral.

\[06.7\] Show that \( x\delta' = \delta \) on \( \mathbb{R} \). Similarly, on \( \mathbb{R}^n \), show that \( x_i \delta = 0 \).

Discussion: These are direct computations, using the characterizations of multiplication and of derivative by duality. For the first assertion, for \( \varphi \in \mathcal{S} \),

\[
(x\delta')(\varphi) = \delta'(x \cdot \varphi) = -\delta((x\varphi)') = -\delta(\varphi + x\varphi') = -\delta(\varphi) + 0 \cdot \varphi'(0) = -\delta(\varphi)
\]

as claimed. On \( \mathbb{R}^n \), for \( \varphi \in \mathcal{S} \),

\[
(x_i \delta)(\varphi) = \delta(x_i \varphi) = 0 \cdot \varphi(0) = 0
\]

as claimed.

\[06.8\] On \( \mathbb{R}^n \), show that \( \Delta \delta = 2n \cdot \delta \).

Discussion: Another direction computation, using the duality characterization: for \( \varphi \in \mathcal{S} \),

\[
(r^2 \Delta \delta)(\varphi) = (\Delta \delta)(r^2 \varphi) = (-1)^2 \delta(\Delta(r^2 \varphi))
\]

Compute

\[
\Delta(r^2 \varphi) = \sum_i \frac{\partial^2}{\partial x_i^2}(r^2 \varphi) = \sum_i \frac{\partial}{\partial x_i}(2x_i \varphi + r^2 \frac{\partial \varphi}{\partial x_i})
\]

\[
= \sum_i 2 \varphi + 2x_i \frac{\partial \varphi}{\partial x_i} + r^2 \frac{\partial^2 \varphi}{\partial x_i^2} = 2n \varphi + \sum_i 2x_i \frac{\partial \varphi}{\partial x_i} + nr^2 \Delta \varphi
\]

Applying \( \delta \) to this gives

\[
2n \varphi(0) + \sum_i 2 \cdot 0 \cdot \frac{\partial \varphi}{\partial x_i}(0) + n \cdot 0 \cdot (\Delta \varphi)(0) = 2n \varphi(0) = 2n \delta(\varphi)
\]

as claimed.

\[06.9\] On \( \mathbb{R}^2 \), compute the Fourier transform of \( (x \pm iy)^n \cdot e^{-\pi(x^2+y^2)} \) for \( n = 0, 1, 2, \ldots \). (Hint: Re-express things, including Fourier transform, in terms of \( z = x + iy \) and \( \overline{z} = x - iy \), \( w = u + iv \), and \( \overline{w} = u - iv \).)

\[1\] This notion of parity can be defined for distributions from the obvious notion for functions \( (\theta \cdot f)(x) = f(-x) \), and then \( (\theta \cdot v)(f) = v(\theta \cdot f) \) for distributions \( v \).
Discussion: Using $z$ and $w$, the functions are $z^n e^{-\pi z \overline{z}}$ and $\overline{z}^n e^{-\pi z \overline{z}}$, and Fourier transform is

$$\int_{\mathbb{R}^2} e^{-\pi i (z \overline{w} + \overline{z} w)} z^n e^{-\pi z \overline{z}} \, dx \, dy = \int_{\mathbb{R}^2} e^{-\pi i (z \overline{w} + \overline{z} w)} \frac{1}{(-\pi)^n} \left( \frac{\partial}{\partial z} \right)^n e^{-\pi z \overline{z}} \, dx \, dy$$

Imagining that we can integrate by parts, this is

$$(-1)^n \frac{1}{(-\pi)^n} \int_{\mathbb{R}^2} \left( \frac{\partial}{\partial z} \right)^n e^{-\pi i (z \overline{w} + \overline{z} w)} e^{-\pi z \overline{z}} \, dx \, dy = \frac{1}{\pi^n} \int_{\mathbb{R}^2} (-\pi i w)^n e^{-\pi i (z \overline{w} + \overline{z} w)} e^{-\pi z \overline{z}} \, dx \, dy$$

$$= (-i)^n w^n \int_{\mathbb{R}^2} e^{-\pi i (z \overline{w} + \overline{z} w)} e^{-\pi z \overline{z}} \, dx \, dy = i^{-n} w^n e^{-\pi (w \overline{w})}$$

since we know the Fourier transform of a Gaussian. A similar computation with roles of $z, \overline{z}$ reversed accomplishes the other computation. That is, $(x \pm iy)^n e^{-\pi (x^2 + y^2)}$ is an eigenfunction for Fourier transform, with eigenvalue $i^{-\lfloor |n| \rfloor}$. //