Examples discussion 09

Paul Garrett  garrett@math.umn.edu  http://www.math.umn.edu/~garrett/

[This document is http://www.math.umn.edu/~garrett/m/real/examples_2017-18/real-disc-09.pdf]

[09.1] Show that the translation action \( T_x f(y) = f(y + x) \) on the Banach space \( C_o^{b,d}(\mathbb{R}) \) of bounded continuous functions on \( \mathbb{R} \) is not continuous. That is, \( \mathbb{R} \times C_o^{b,d}(\mathbb{R}) \to C_o^{b,d}(\mathbb{R}) \) by \( x \to T_x f \) is not continuous. In particular, find a particular \( f \in C_o^{b,d}(\mathbb{R}) \) with \( |f|_{C^o} = 1 \) such that, there is a sequence \( \delta_n \to 0 \) of non-zero numbers \( \delta_n \) such that \( |T_{\delta_n} f - f|_{C^o} = 1 \).

Discussion: The point is that on a non-compact topological space there may exist continuous, bounded, but not uniformly continuous functions, such as \( f(x) = \sin(x^2) \). Let \( x_n = n \cdot \sqrt{\pi} \) and let \( \delta_n > 0 \) be a sequence of small positive reals going to 0 such that \( (x_n + \delta_n)^2 = x_n^2 + \frac{\pi}{2^2} \). Then \( \sin(x_n^2) = 0 \), while \( \sin((x_n + \delta_n)^2) = 1 \), so the sup norm of \( \sin(x^2) - \sin((x + \delta_n)^2) \) is 1.

[9.1] Remark: Nevertheless, the translation action is continuous on \( C_o^{b,d}(\mathbb{R}) \), which we see as follows. Given \( f \in C_o^{b,d}(\mathbb{R}) \), for given \( \varepsilon > 0 \), by a previous example there is \( g \in C_o^{b,d}(\mathbb{R}) \) such that \( \sup x \in \mathbb{R} |g(x) - f(x)| < \varepsilon \).

Since \( g \) is compactly supported, it is uniformly continuous, so there is \( \delta > 0 \) such that \( |x - y| < \delta \) implies \( |g(x) - g(y)| < \varepsilon \). Then for \( |h| < \delta \),

\[
\sup_{x \in \mathbb{R}} |f(x + h) - f(x)| \leq \sup_{x \in \mathbb{R}} |f(x + h) - g(x + h) - (f(x) - g(x))| + \sup_{x \in \mathbb{R}} |g(x + h) - g(x)|
\]

\[
\leq \sup_{x \in \mathbb{R}} |f(x + h) - g(x + h)| + \sup_{x \in \mathbb{R}} |f(x) - g(x)| + \sup_{x \in \mathbb{R}} |g(x + h) - g(x)| < \varepsilon + \varepsilon + \varepsilon
\]

This is half the desired continuity, in contrast to the problem with \( C_o^{b,d}(\mathbb{R}) \). Similarly, the translation action \( \mathbb{R} \times C_o^{b,d}(\mathbb{R}) \) is jointly continuous in both arguments.

[09.2] Let \( r_1, r_2, r_3, \ldots \) be an enumeration of the rational numbers inside the interval \([0, 1]\). Define \( T : l^2 \to l^2 \) by \( T(r_1, c_2, \ldots) = (r_1 c_1, r_2 c_2, \ldots) \). Show that \( T \) is a continuous/bounded linear operator, is self-adjoint, has eigenvalues exactly the \( r_1, r_2, r_3, \ldots \), and continuous spectrum the whole interval \([0, 1]\) (with rationals removed, if one insists on disjointness of discrete and continuous spectrum).

Discussion: Since the set \( \{|r_1|, |r_2|, \ldots\} \) is bounded by 1, the operator norm of \( T \) is at most 1, so it is bounded, hence continuous. Since the \( r_n \) are all real, the operator is self-adjoint:

\[
\langle T(a_1, a_2, \ldots), (b_1, b_2, \ldots) \rangle = \langle (r_1 a_1, r_2 a_2, \ldots), (b_1, b_2, \ldots) \rangle = \sum_n r_n a_n \cdot \overline{b_n}
\]

When \( \lambda \cdot (c_1, c_2, \ldots) = T(c_1, c_2, \ldots) = (r_1 c_1, r_2 c_2, \ldots) \), necessarily \( \lambda \cdot c_n = r_n \cdot c_n \) for all \( n \). When \( c_n \neq 0 \), this implies \( \lambda = r_n \). Since the \( r_n \) are distinct, there can be (at most) one index \( n \) for which \( c_n \neq 0 \), and then \( \lambda = r_n \). Conversely, every \( r_n \) is obviously an eigenvalue.

Since we know that the whole spectrum is closed in \( \mathbb{C} \), it contains at least the closure of the rationals in \([0, 1]\), namely, \([0, 1]\) itself. Since \( T \) is self-adjoint, its spectrum is contained in \( \mathbb{R} \). [1] Since the spectrum is bounded by \( |T|_{op} = 1 \), it is contained in \([-1, 1]\).

[1] The proof that self-adjoint operators \( T \) have spectrum inside \( \mathbb{R} \) has more content than just the analogous assertion about eigenvectors. For \( T v = \lambda v \) with \( v \neq 0 \), of course

\[
\lambda(v, v) = \langle \lambda v, v \rangle = \langle T v, v \rangle = \langle v, T v \rangle = \langle v, \lambda v \rangle = \overline{\lambda(v, v)}
\]

shows that any eigenvalues are real. Since self-adjoint operators have no residual spectrum, to find the rest of the
To see that \( \lambda \in [-1, 0) \) is not in the spectrum, in that \( (T - \lambda)(c_1, c_2, \ldots) = ((r_1 - \lambda)c_1, (r_2 - \lambda)c_2, \ldots) \), we have \( |r_n - \lambda| \geq |\lambda| > 0 \), so the inverse \( (T - \lambda)^{-1} \) can be written down immediately: \( (T - \lambda)^{-1}(c_1, c_2, \ldots) = ((r_1 - \lambda)^{-1}c_1, (r_2 - \lambda)^{-1}c_2, \ldots) \) and there is a uniform upper bound \( |(r_n - \lambda)^{-1}| \leq |\lambda|^{-1} \). Finally, given irrational \( \lambda \in [0, 1) \), let \( r_{n_1}, r_{n_2}, \ldots \) be rationals such that \( r_{n_i} \to \lambda \). With standard basis \( \{e_n\} \) for \( \ell^2 \), we claim that \( \{e_{n_i}\} \) is an approximate eigenvector for \( \lambda \): given \( \varepsilon > 0 \), let \( N \) be sufficiently large so that \( |r_{n_i} - \lambda| < \varepsilon \) for \( i \geq N \). For \( n_i \geq N \),

\[
|(T - \lambda)e_{n_i}| = |(r_{n_i} - \lambda)e_{n_i}| = |r_{n_i} - \lambda| |e_{n_i}| = |r_{n_i} - \lambda| < \varepsilon
\]

Thus, indeed, \( (T - \lambda)e_{n_i} \to 0 \), and the \( e_{n_i} \) give an approximate identity for \( \lambda \), so \( \lambda \) is in the spectrum.

//

9.3 Let \( r_1, r_2, r_3, \ldots \) be a bounded sequence of complex numbers. Define \( T : \ell^2 \to \ell^2 \) by \( T(c_1, c_2, \ldots) = (r_1c_1, r_2c_2, \ldots) \). Show that \( T \) is compact if and only if \( r_n \to 0 \).

Discussion: Let \( e_1, e_2, \ldots \) be the standard (Hilbert-space) basis for \( \ell^2 \). If the \( r_n \) do not go to 0, then there is a subsequence \( r_{n_1}, r_{n_2}, \ldots \) bounded away from 0. Since \( T \) is compact, the images \( Te_{n_i} = r_{n_i}e_{n_i} \) must have a convergent subsequence. But \( |r_{n_i}e_{n_i} - r_{n_j}e_{n_j}|^2 = |r_{n_i}|^2 + |r_{n_j}|^2 \) for \( i \neq j \), and this is bounded away from 0, so there is no convergent subsequence, contradicting the compactness of \( T \). Thus, in fact, \( r_n \to 0 \).

For the converse, perhaps the most economical approach is to observe that \( T \) is an operator-norm limit of finite-rank operators, hence compact:

\[
T_n(c_1, c_2, \ldots, c_n, c_{n+1}, \ldots) = (c_1, c_2, \ldots, c_n, 0, 0, \ldots)
\]

The estimate on the operator norms is

\[
|T - T_n|_{op} = \sup_{|v| = 1} |(0, 0, \ldots, 0, 0, 0, \ldots)| = \sup_{k \geq n} |r_k| \to 0
\]

Less efficiently, we can refer to definitions, and use the total boundedness criterion for compact closure. Given \( \varepsilon > 0 \), let \( N \) be large enough so that \( |r_n| < \varepsilon \) for \( n \geq N \). Write \( v = (v_1, v_2, \ldots) \in \ell^2 \) as

\[
v = \underbrace{(v_1, \ldots, v_N, 0, 0, \ldots)}_{v'} + \underbrace{(0, \ldots, 0, v_{N+1}, \ldots, v_{N+2}, \ldots)}_{N}
\]

Let \( B' \) be the intersection of the unit ball \( B \subset \ell^2 \) with the copy of \( \mathbb{C}^N \subset \ell^2 \) with non-zero components only at the first \( N \) places. Let \( B'' \) be the intersection of \( B \) with the subspace of \( \ell^2 \) with 0 entries at the first \( N \) places. Certainly \( B' + B'' \supset B \) and \( B' \perp B'' \).

By design, \( |Tv''| \leq \varepsilon \) for \( v'' \in B'' \). Since \( TB' \) is a bounded subset of a finite-dimensional space \( \mathbb{C}^N \), it has compact closure, so is totally bounded, so can be covered by finitely-many \( \varepsilon \)-balls \( U_1, \ldots, U_k \). Then \( TB \subset TB' + TB'' \subset \bigcup (U_1 + TB'') \cup \ldots \cup (U_k + TB'') \), and every \( U_i + TB'' \) is contained in a 2\( \varepsilon \)-ball. Thus, \( TB \) is totally bounded, hence, has compact closure. //

Spectrum it suffices to identify approximate eigenvectors. Note that for self-adjoint \( T \) always \( \langle Tv, v \rangle = \langle v, Tv \rangle \), so \( \langle Tv, v \rangle \) is real. Then for \( (T - \lambda)v \to 0 \), certainly \( \langle (T - \lambda)v_n, v_n \rangle \to 0 \), so the imaginary parts go to 0. These are

\[
\text{Im} \langle (T - \lambda)v_n, v_n \rangle = \text{Im} \langle Tv_n, v_n \rangle + \text{Im} \langle \lambda \cdot (v_n, v_n) \rangle = 0 + \text{Im} \langle \lambda \cdot (v_n, v_n) \rangle
\]

Since \( |v_n| \) are bounded away from 0, there can be an approximate identity only for \( \lambda \in \mathbb{R} \). //

[2] For such a simple operator, a similar device shows that \( \lambda \notin \mathbb{R} \) is not in the spectrum.
[09.4] Let $T$ be a compact operator $T : V \to W$ for Hilbert spaces $V, W$. For $S$ a continuous/bounded operator on $V$, show that $T \circ S : V \to W$ is compact. For $R$ a continuous/bounded operator on $W$, show that $R \circ T : V \to W$ is compact.

**Discussion:** For $T \circ S$, the image of the unit ball under $S$ is contained in some ball $c \cdot B$, where $B$ is the unit ball, because $S$ is bounded. Since $T$ is linear, $T(c \cdot B) = c \cdot TB$. Since $TB$ is pre-compact, its continuous image under multiplication by $c$ is also pre-compact. Proof: for $c = 0$, we’re done. For $c > 0$, given a finite cover of $TB$ by balls $w_i + B_e$ where $B_e$ is the ball of radius $\varepsilon > 0$ centered at 0. The images $c \cdot (w_i + B_e) = cw_i + cB_e$ cover $c \cdot TB$, and have radius $c \cdot \varepsilon$. Replacing $\varepsilon$ by $\varepsilon/c$ gives balls of radius $\varepsilon$ covering $c \cdot TB$.

For $R \circ T$, similarly as in the previous case, given a finite cover of $TB$ by balls $w_i + B_e$ of radius $\varepsilon > 0$, the images $R(w_i + B_e) = Rw_i + RB_e$ are contained in balls $Rw_i + cB_e$, where $c = |R|_{op}$ will suffice. ///

[09.5] Let $S, T$ be two compact, self-adjoint operators on a Hilbert space, and $ST = TS$. Show that there is an orthonormal basis for $V$ consisting of simultaneous eigenfunctions for $S, T$.

**Discussion:** The Hilbert space $V$ is the closure of the orthogonal direct sum of eigenspaces $V_\lambda$ for $T$. For $\lambda \neq 0$, $V_\lambda$ is finite-dimensional, so is necessarily closed, and $V_0$ is the orthogonal complement of the sum of all other eigenspaces, so is closed. Since $ST = TS$, we find that $S$ stabilizes each $V_\lambda$:

$$T(Sv) = (TS)v = (ST)v = S(Tv) = S(\lambda v) = \lambda \cdot Sv \quad \text{(for all } v \in V_\lambda)$$

[9.2] Claim: The restriction of a compact operator to a closed subspace $W \subset V$ stabilized by it is still compact.

**Proof:** With $B'$ the closed unit ball of $W$ and $B$ the closed unit ball of $V$, $TB' \subset TB$. Using the total-boundedness criterion for precompactness, given $\varepsilon > 0$, $TB$ is covered by finitely-many $\varepsilon$-balls $v_i + B_e$. Among the intersections $W \cap (v_i + B_e)$, the non-empty ones are open balls of radius at most $\varepsilon$. Thus, $TB'$ is a precompact set, and $T|_W$ is a compact operator. ///

Thus, $S$ is a compact operator on each $V_\lambda$, so every $V_\lambda$ has an orthonormal basis of $S$-eigenvectors. These are also $\lambda$-eigenvectors for $T$, so they are simultaneous eigenvectors. ///

[09.6] Recall the proof that the Hilbert cube

$$C = \{(z_1, z_2, \ldots) \in \ell^2 : |z_n| \leq \frac{1}{n}\}$$

is compact. More generally, for any sequence of positive reals $r_n$,

$$C(r) = \{(z_1, z_2, \ldots) \in \ell^2 : |z_n| \leq r_n\}$$

is compact if and only if $\sum_n |r_n|^2 < \infty$.

**Discussion:** Use the total boundedness criterion. Given $\varepsilon > 0$, by convergence of $\sum_n \delta_n^2$, there is $n_0$ large enough so that $\sum_{n \geq n_0} r_n^2 < \varepsilon^2$. The set

$$C_{n_0} = \{(z_1, z_2, \ldots, z_{n_0}) \in \mathbb{R}^{n_0} : |z_n| \leq r_n\}$$

is a compact subset of $\mathbb{C}^{n_0}$, so certainly has a finite cover by open balls of radius $\varepsilon$. Let the centers of these balls be $w_1, \ldots, w_N$. Let $j : \mathbb{C}^{n_0} \to \ell^2$ be the inclusion $j(z_1, \ldots, z_{n_0}) = (z_1, \ldots, z_{n_0}, 0, 0, \ldots)$. Then we claim that the open balls of radius $2\varepsilon$ at $j(w_1), j(w_2), \ldots, j(w_N)$ cover $C(r)$. Indeed, given $z = (z_1, z_2, \ldots) \in C(r)$,
write $z = j(z') + z''$ where $z' = (z_1, \ldots, z_{n_0})$ and $z'' = z - j(z') = (0, \ldots, 0, z_{n_0+1}, \ldots)$. There is at least one of the $w_j$s within $\varepsilon$ of $z'$; let $w_{j_0}$ be such. By the triangle inequality for the norm $| \cdot |_{L^2}$ on $\ell^2$, 
\[
d(z, j(w_{j_0})) = |z - j(w_{j_0})|_{L^2} = |j(z') + z'' - j(w_{j_0})|_{L^2} \leq |j(z') - j(w_{j_0})|_{L^2} + |z''|_{L^2} = |z' - w_{j_0}|_{\mathbb{R}^{n_0}} + |z''|_{L^2} < \varepsilon + \varepsilon
\]
Thus, $C(r)$ can be covered by finitely-many open balls of radius $2\varepsilon$.

[09.7] First, for Schwartz $\varphi$ on $\mathbb{R}^n$ and $u$ a tempered distribution on $\mathbb{R}^n$, characterize $\varphi * u$. Show that $\varphi \hat{*} \check{u} = \hat{\varphi} \cdot \check{u}$, where the latter multiplication is that induced by duality: $(\hat{\varphi} \cdot \check{u})(\psi) = \hat{u}(\hat{\varphi} \cdot \psi)$ for $\psi \in \mathcal{S}$. Explain why the union $H^{-\infty}$ of Sobolev spaces is inside the space of tempered distributions, and why $\hat{u}$ has pointwise values for $u \in H^{-\infty}$.

Discussion: To anticipate a characterization of $\varphi * u$, we can examine $\varphi * u_f$ where $u_f$ is integrate-against (for example) a locally integrable function of moderate growth, since the characterization for tempered distributions should extend (continuously...) that for distributions given by integrate-against-functions. Writing $f^\theta(x) = f(-x)$ to avoid confusion with Fourier transform notations, for $\psi \in \mathcal{S}$, invoking Fubini-Tonelli as needed to change order of integration,
\[
(\varphi * u_f)(\psi) = \int f (\varphi * u_f) \psi = \int \int \varphi(x - y) f(y) \psi(x) \, dy \, dx = \int \int \varphi^\theta(y - x) f(y) \psi(x) \, dx \, dy
\]
\[
= \int (\varphi^\theta * \psi)(y) f(y) \, dy \, dx = u_f(\varphi^\theta * \psi)
\]
It is important to note that $\varphi^\theta * \psi$ (with or without the $\theta$) is still a Schwartz function. (One might reflect on the easiest way to be sure of this...) Thus, we can specify the tempered distribution $\varphi * u$ by $(\varphi * u)(\psi) = u(\varphi^\theta * \psi)$.

Since this extends the corresponding operation on distributions given by integration-against functions, but we can check once-again via this definition: for $\alpha, \beta \in \mathcal{S}$,
\[
(\alpha * (\beta * u))(\psi) = (\beta * u)(\alpha^\theta * \psi) = u(\beta^\theta * (\alpha^\theta * \psi)) = u((\beta^\theta * \alpha^\theta) * \psi)
\]
by associativity of convolution on Schwartz functions, which by elementary (change-of-variables) properties of $\theta$ is
\[
u((\alpha * \beta)^\theta) * \psi) = ((\alpha * \beta) * u)(\psi)
\]
proving the associativity.

Letting $F$ denote Fourier transform when notationally convenient,
\[
\hat{\varphi} * u(\psi) = (\varphi * u)(\hat{\psi}) = u(\varphi^\theta * \hat{\psi}) = u(F(\hat{\varphi} \cdot \psi))
\]
since $\varphi^\sim = \varphi^\theta$. This is
\[
\hat{u}(\hat{\varphi} \cdot \psi) = (\hat{\varphi} \cdot \check{u})(\psi)
\]
by the definition of multiplication of tempered distributions by Schwartz functions, extending pointwise multiplication.

Since Fourier transform maps $\mathcal{S}$ isomorphically to itself, and since $\mathcal{S}$ is certainly inside all the weighted $L^2$ spaces used to define the Sobolev spaces $H^s$, we have $\mathcal{S} \subset H^\infty$.

Since $H^\infty \subset \mathcal{E} = C^\infty$ by Sobolev imbedding, taking duals gives $\mathcal{E}^* \subset (H^\infty)^* = H^{-\infty}$. In particular, since distributions in $H^{-\infty}$ have Fourier transforms in weighted $L^2$ spaces, hence have pointwise values almost-everywhere, compactly-supported distributions have Fourier transforms with pointwise values almost-everywhere. (In fact, there is a Paley-Wiener theorem for compactly-supported distributions, due to L. Schwartz.)
Thus, for $u \in \mathcal{E}^*$, or even $u \in H^{-\infty}$, for $\varphi \in \mathcal{S}$, the Fourier transform $\hat{\varphi} \ast \hat{u} = \hat{\varphi} \cdot \hat{u}$ has pointwise values almost-everywhere, and thus it makes sense to assert that

$$\int_{\mathbb{R}} |\hat{\varphi} \ast \hat{u}|^2 = \int_{\mathbb{R}} |\hat{\varphi} \cdot \hat{u}|^2$$