For feedback on these examples, please get your write-ups to me by Monday, 25 Sept, 2017.

[01.1] Prove that every open in \( \mathbb{R}^2 \) is a countable union of Cartesian products \((a, b) \times (c, d)\) of open intervals.

[01.2] Define tent functions of width \( w \) and height \( h \) centered at 0 by

\[
t_{w,h}(x) = \begin{cases} 0 & \text{(for } x \leq -w) \\ \frac{h}{w} \cdot (x + w) & \text{(for } -w \leq x \leq 0) \\ h - \frac{h}{w} \cdot x & \text{(for } 0 \leq x \leq w) \\ 0 & \text{(for } x \geq w) \end{cases}
\]

Show that the functions \( f_n(x) = t_{\frac{1}{n},n}(x - \frac{1}{n}) \), a sequence of narrowing tents just to the right of 0, go to 0 pointwise (everywhere!), but that

\[
\lim_{n} \int_{\mathbb{R}} f_n(x) \cdot g(x) \, dx = g(0) \quad \text{(for all } g \in C^o(\mathbb{R}))
\]

[01.3] Show that the functions \( f_n(x) = t_{\frac{1}{n},n^2}(x - \frac{1}{n}) - t_{\frac{1}{n},n^2}(x + \frac{1}{n}) \), whose graphs are tall tents of area \( n \) upward just to the right of 0, and tall tents downward just to the left of 0, go to 0 pointwise everywhere, but that

\[
\lim_{n} \int_{\mathbb{R}} f_n(x) \cdot g(x) \, dx = 2g'(0) \quad \text{(for differentiable } g \text{ with derivative } g' \text{ in } C^o(\mathbb{R}))
\]

[01.4] Show that the closed unit ball in \( \ell^2 \), although closed and bounded, is not compact, by showing it is not sequentially compact.

[01.5] Show that the Hilbert cube

\[ C = \{(z_1, z_2, \ldots) \in \ell^2 : |z_n| \leq \frac{1}{n}\} \]

is compact. More generally, for any sequence of positive reals \( \varepsilon_n \),

\[ C(\varepsilon) = \{(z_1, z_2, \ldots) \in \ell^2 : |z_n| \leq \varepsilon_n\} \]

is compact if and only if \( \sum_{n} |\varepsilon_n|^2 < \infty \).

[01.6] Show that a closed interval \([a, b]\) has the expected Lebesgue outer measure, namely, \(|b-a|\), by showing that the inf of \( \sum_{j=1}^{n} |b_n - a_n| \) for all finite open covers \([a, b] \subset \bigcup_{j=1}^{n} (a_j, b_j)\) is \(|b-a|\).

[01.7] Let \( f \) be a continuous function on \([0,1]\), with \( f(0) = 0 \) and \( f(1) = 1 \). Show that \( \{x : f(x) \in [\frac{1}{4}, \frac{3}{4}]\} \) has positive Lebesgue measure.