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Examples 02

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[This document is http://www.math.umn.edu/~garrett/m/real/examples_2017-18/real-ex-02.pdf]

For feedback on these examples, please get your write-ups to me by Monday, 09 Oct 2017.

[02.1] Show that $\ell^2$ is complete as a metric space.

[02.2] Show that the characteristic function $\chi_E$ of a measurable set $E$ is measurable.

[02.3] Show that the product of two $\mathbb{R}$-valued measurable functions on $\mathbb{R}$ is measurable.

[02.4] Use Urysohn’s lemma to prove that $C^0[a,b]$ is dense in $L^1[a,b]$.

[02.5] Comparing $L^p$ spaces: let $1 \leq p, p' < \infty$. When is $L^p[a,b] \subset L^{p'}[a,b]$ for finite intervals $[a,b]$ and Lebesgue measure? When is $L^p(\mathbb{R}) \subset L^{p'}(\mathbb{R})$? When is $\ell^p \subset \ell^{p'}$?

[02.6] For positive real numbers $w_1, \ldots, w_n$ such that $\sum_i w_i = 1$, and for positive real numbers $a_1, \ldots, a_n$, show that

$$a_1^{w_1} \ldots a_n^{w_n} \leq w_1 a_1 + \ldots + w_n a_n$$

[02.7] In $\ell^2$, show that the point in the closed unit ball closest to a point $v$ not inside that ball is $v/|v|\ell^2$.

[02.8] For a measurable set $E \subset [0,2\pi]$, show that

$$\lim_{n \to \infty} \int_E \cos nx \, dx = 0 = \lim_{n \to \infty} \int_E \sin nx \, dx$$

[02.9] One form of the sawtooth function is $f(x) = x - \pi$ on $[0,2\pi]$. Compute the Fourier coefficients $\hat{f}(n)$. Write out the conclusion of Plancherel-Parseval’s theorem for this function.

[02.10] For fixed $y \in [0,2\pi]$, show that there is no $f_y \in L^2[0,2\pi]$ so that $\langle g, f_y \rangle = g(y)$ for all $g \in L^2[0,2\pi]$. 