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Examples 03

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[This document is
http://www.math.umn.edu/~garrett/m/real/examples_2017-18/real-ex-03.pdf]

For feedback on these examples, please get your write-ups to me by Wednesday, 18 Oct 2017.

[03.1] Show that every vector subspace of \( \mathbb{R}^n \) and/or \( \mathbb{C}^n \) is (topologically) closed.

[03.2] For a subspace \( W \) of a Hilbert space \( V \), show that \((W^\perp)^\perp\) is the closure of the subspace \( W \) in \( V \).

[03.3] Show that for \( 0 < x < 1 \)
\[ \sum_{n \geq 1} \frac{\sin 2\pi nx}{n} = \pi \cdot \left( \frac{1}{2} - x \right) \]

[03.4] Let \( c_1, c_2, \ldots \) be positive real, converging monotonically to 0. For \( 0 < x < 1 \), prove that \( \sum_{n \geq 0} c_n e^{2\pi inx} \) converges pointwise.

[03.5] Show that the sup-norm completion of the space \( C_0^c(\mathbb{R}) \) of compactly-supported continuous functions is the space \( C_0^\infty(\mathbb{R}) \) of continuous functions going to 0 at infinity. An analogous assertion and argument should hold for any topological space in place of \( \mathbb{R} \).

[03.6] Compute \( \int_{\mathbb{R}} \left( \frac{\sin x}{x} \right)^2 \, dx \). (Hint: use Plancherel.)

[03.7] For \( f \in L^2(\mathbb{R}) \) and \( t \in \mathbb{R} \), show that there is a constant \( C \) (depending on \( f \)) such that
\[ \left| \int_{t-\delta}^{t+\delta} f(x) \, dx \right| < C \cdot \sqrt{\delta} \]
Formulate and prove the corresponding assertion for \( L^p \) with \( 1 < p < \infty \).

[03.8] For \( f \in L^1(\mathbb{R}) \) and \( t \in \mathbb{R} \), show that, given \( \varepsilon > 0 \), there is \( \delta > 0 \) such that
\[ \left| \int_{t-\delta}^{t+\delta} f(x) \, dx \right| < \varepsilon \]
Sharpen the first example to show that
\[ \int_{t-\delta}^{t+\delta} f(x) \, dx = o(\sqrt{\delta}) \quad \text{ as } \delta \to 0^+ \]
where Landau’s little-o notation is that \( f(x) = o(g(x)) \) as \( x \to a \) when \( \lim_{x \to a} f(x)/g(x) = 0 \).