Examples 04

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[This document is http://www.math.umn.edu/~garrett/m/real/examples_2017-18/real-ex-04.pdf]

For feedback on these examples, please get your write-ups to me by Monday, 30 Oct 2017.

[04.1] Fix \(x_0 \in [a,b]\). Show that \(\lambda(f) = f(x_0)\) is a continuous linear functional on \(C^0[a,b]\).

[04.2] Prove that Cesaro summation

\[
b_1 = \frac{a_1}{1}, \quad b_2 = \frac{a_1 + a_2}{2}, \quad b_3 = \frac{a_1 + a_2 + a_3}{3}, \ldots
\]

converts every convergent sequence \(a_1, a_2, \ldots\) to a convergent sequence \(b_1, b_2, \ldots\) with the same limit.

[04.3] (Collecting Fourier transform pairs...) Compute the Fourier transforms of

\[
\chi_{[a,b]} = e^{-\pi x^2}, \quad f(x) = \begin{cases} e^{-x} & \text{(for } x > 0) \\ 0 & \text{(for } x \leq 0) \end{cases}
\]

[04.4] Show that \(\chi_{[a,b]} * \chi_{[c,d]}\) is a piecewise-line function, and express it explicitly.

[04.5] Evaluate the Borwein integral

\[
\int_{\mathbb{R}} \frac{\sin x}{x} \cdot \frac{\sin x/3}{x/3} \cdot \frac{\sin x/5}{x/5} \, dx
\]

[04.6] Compute \(e^{-\pi x^2} * e^{-\pi x^2}\) and \(\frac{\sin x}{x} * \frac{\sin x}{x}\). (Be careful what you say: \(\frac{\sin x}{x}\) is not in \(L^1(\mathbb{R})\).)

[04.7] Prove that every \(f \in C^0_c(\mathbb{R})\) can be uniformly approximated (in sup norm) arbitrarily well by superpositions of translates of Gaussians: given \(\varepsilon > 0\), there is \(\varphi \in C^0_c(\mathbb{R})\) and sufficiently large \(n\) such that

\[
\sup_{x \in \mathbb{R}} \left| f(x) - \int_{\mathbb{R}} \varphi(\xi) \cdot e^{-\pi n^2(\xi - x)^2} \, d\xi \right| < \varepsilon
\]

[04.8] Prove that, for given \(f \in C^0_c(\mathbb{R})\), given \(\varepsilon > 0\), there is sufficiently large \(n\) and a function \(\varphi \in C^0_c(\mathbb{R})\) such that

\[
\sup_{x \in \mathbb{R}} \left| f(x) - \int_{\mathbb{R}} \varphi(\xi) \cdot \frac{(\sin n(x - \xi))^2}{(x - \xi)^2} \, d\xi \right| < \varepsilon
\]

[04.9] Show that the principal value functional

\[
f \longrightarrow PV \int_{\mathbb{R}} \frac{f(x)}{x} \, dx = \lim_{\varepsilon \to 0} \left( \int_{-\infty}^{-\varepsilon} \frac{f(x)}{x} \, dx + \int_{\varepsilon}^{\infty} \frac{f(x)}{x} \, dx \right)
\]
is equal to

\[-\int_{\mathbb{R}} f'(x) \cdot \log |x| \, dx\]

for \( f \) continuously differentiable near 0, with \( f \in L^2(\mathbb{R}) \), \( f(x) \to 0 \) as \( |x| \to \infty \), and \( |f'(x)| \ll \frac{1}{1+x^2} \) for easy convergence.

\[\textbf{[04.10]}\] Let \( \psi_n(x) = e^{2\pi i nx} \). Let \( \delta_Z \) be the Dirac comb, that is, a periodic version of Dirac’s \( \delta \), describable as having Fourier series

\[\delta_Z = \sum_{n \in \mathbb{Z}} 1 \cdot \psi_n \quad \text{(converging in } H^{-1}(T) \text{ or even } H^{-\frac{1}{2}-\varepsilon}(T) \text{ for all } \varepsilon > 0)\]

With \( \lambda \not\in \mathbb{R} \), show that the differential equation

\[u'' - \lambda \cdot u = \delta_Z\]

has a periodic solution \( u \in H^{\frac{1}{2}-\varepsilon}(T) \subset C^\alpha(T) \), using Fourier series, by division. Show that the equation \( v'' - \lambda v = f \) is solved by

\[v = \int_T u(x-t) \, f(t) \, dt = \int_0^1 u(x-t) \, f(t) \, dt\]