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Examples 05

Paul Garrett  garrett@math.umn.edu  http://www.math.umn.edu/~garrett/

For feedback on these examples, please get your write-ups to me by Wednesday, 15 Nov 2017.

[05.1] Give a persuasive proof that the function
\[ f(x) = \begin{cases} 
0 & \text{for } x \leq 0 \\
\exp(-1/x) & \text{for } x > 0
\end{cases} \]
is infinitely differentiable at 0. Use this kind of construction to make a smooth step function: 0 for \( x \leq 0 \) and 1 for \( x \geq 1 \), and goes monotonically from 0 to 1 in the interval \([0,1]\). Use this to construct a family of smooth cut-off functions \( \{ f_n : n = 1, 2, 3, \ldots \} \): for each \( n \), \( f_n(x) = 1 \) for \( x \in [-n,n] \), \( f_n(x) = 0 \) for \( x \not\in [-n+1,n+1] \), and \( f_n \) goes monotonically from 0 to 1 in \([-n+1,-n]\) and monotonically from 1 to 0 in \([n,n+1]\).

[05.2] With \( g(x) = f(x + x_0) \), express \( \tilde{g} \) in terms of \( \hat{f} \), first for \( f \in \mathcal{S}((\mathbb{R}^n)^*) \), then for \( f \in \mathcal{S}((\mathbb{R}^n)^*)^* \).

[05.3] Let \( V \) be a vector space, with norms \( |\cdot|_1 \) and \( |\cdot|_2 \). Suppose that \( |v|_2 \geq |v|_1 \) for all \( v \in V \). Show that the identity map \( i : V \to V \) is continuous, where the source is given the \( |\cdot|_2 \) topology and the target is given the \( |\cdot|_1 \) topology. Show that if a sequence \( \{ v_n \} \) in \( V \) is \( |\cdot|_2 \)-Cauchy, then it is \( |\cdot|_1 \)-Cauchy. Let \( V_j \) be the completion of \( V \) with respect to the metric \( |v - v'|_j \). Show that we can extend \( i \) by continuity to a continuous linear map \( I : V_2 \to V_1 \), that is, by

\[ I(V_2 \text{-limit of } V_2 \text{-Cauchy sequence } \{ v_n \}) = V_1 \text{-limit of } \{ v_n \} \]

[05.4] Solve \(-u'' + u = \delta \) on \( \mathbb{R} \). (Hint: use Fourier transform, and grant that \( \hat{\delta} = 1 \).)

[05.5] Show that \( u'' = \delta_Z \) has no solution on the circle \( \mathbb{T} \). (Hint: Use Fourier series, granting the Fourier expansion of \( \delta_Z \).) Show that \( u'' = \delta_Z - 1 \) does have a solution.

[05.6] On the circle \( \mathbb{T} \), show that \( u'' = f \) has a unique solution for all \( f \in L^2(\mathbb{T}) \) orthogonal to the constant function 1.

[05.7] The sawtooth function is first defined on \([0,1]\) by \( \sigma(x) = x - \lfloor \frac{x}{\frac{1}{2}} \rfloor \), and then extended to \( \mathbb{R} \) by periodicity so that \( \sigma(x+n) = \sigma(x) \) for all \( x \in \mathbb{R} \) and \( n \in \mathbb{Z} \). After recalling its Fourier expansion, describe the derivatives \( \sigma' \) and \( \sigma'' \) of \( \sigma \).

[05.8] Show that \( e^{-\varepsilon x^2} \to 1 \) as \( \varepsilon \to 0^+ \) in the \( \mathcal{S}^* \) topology. Compute the Fourier transforms of the functions \( e^{-\varepsilon x^2} \), and show that they go to \( \delta \) in the \( \mathcal{S}^* \) topology. Obtain, again, as a corollary, the fact that \( \hat{1} = \delta \) (extended Fourier transform).

[05.9] Compute \( \widehat{\cos x} \). (Hint: write \( \cos x \) in terms of complex exponentials, and observe that these complex exponentials are the Fourier transforms of certain translates of \( \delta \).)

[05.10] Smooth functions \( f \in \mathcal{E} \) act on distributions \( u \in \mathcal{D}(\mathbb{R})^* \) by a dualized form of pointwise multiplication: \( (f \cdot u)(\varphi) = u(f \varphi) \) for \( \varphi \in \mathcal{D}(\mathbb{R}) \). Show that if \( x \cdot u = 0 \), then \( u \) is supported at 0, in the sense that for \( \varphi \in \mathcal{D} \) with spt \( \varphi \not\in 0 \), necessarily \( u(\varphi) = 0 \). Thus, by the theorem classifying such distributions, \( u \) is a linear combination of \( \delta \) and its derivatives. Show that in fact \( x \cdot u = 0 \) implies that \( u \) is a multiple of \( \delta \) itself.