[01.1] Show that the closed unit ball in $\ell^2$, although closed and bounded, is not compact, by showing it is not sequentially compact.

Discussion: Let $e_n = (0, \ldots, 0, 1, 0, \ldots)$ with the single 1 at the $n^{th}$ place. Then $d(e_m, e_n) = \sqrt{2}$ for $m \neq n$. Thus, the sequence of $e_n$’s has no Cauchy subsequence, so no convergent subsequence. ///

[01.2] Show that the closed unit ball in $C^0[a, b]$ is not compact, despite being closed and bounded.

Discussion: Let $f_n$ be a tent function centered at $1/2^n$, of height 1, and width $1/2^{n+2}$ (or anything strictly larger than $1/2^{n+1}$). By design, the supports of these functions are disjoint, and all their sup-norms are 1. Thus, for $m \neq n$, $|f_m - f_n|_{C^0} = 1$. Thus, the sequence has no Cauchy subsequence. ///

[01.3] Let $X$ be a metric space with a countable dense subset $D$. Show that every open set in $X$ is a countable union of open balls.

Discussion: Let $U$ be the open set. For $x \in U$, let $B(r_x, x)$ be an open ball or radius $r_x$ centered at $x$ and contained in $U$. We can shrink $r_x$ to make it rational. By density, there is an element $d_x$ in the smaller ball $B(r_x/2, x)$. Then $B(r_x/2, d_x)$ contains $x$ and is inside $B(r_x, x)$, so is inside $U$. Thus, $U \subset \bigcup_{x \in D} B(r_x/2, d_x)$. By countability of $D$ and of rationals (the radii), there can be only countably-many distinct balls $B(r_x, d_x)$. ///

[01.4] Let $X$ be a compact metric space. Show that a continuous function on $X$ is uniformly continuous.

Discussion: Let $f \in C^0(X)$. Given $\varepsilon > 0$, for each $x \in X$ let $B(r_x, x)$ be a ball of radius $r_x$ centered at $x$ such that $|f(x) - f(y)| < \varepsilon$ for $y \in B(r_x, x)$. The open sets $B(r_x/2, x)$ cover $X$. By compactness, there is a finite subcover $B(r_x/2, x_1), \ldots, B(r_x/2, x_n)$. Thus, given $y, z \in X$ with $d(y, z) < \min_i r_x/2$, let $y \in B(r_x/2, x_i)$. Then $z \in B(r_x, x_i)$, as is $y$. Thus, $|f(y) - f(z)| < \varepsilon$. ///

[01.5] Let $X$ be a compact metric space. Show that a uniform pointwise limit of continuous real-valued functions is continuous.

Discussion: This is a slightly abstracted version of the iconic three-epsilon argument. Let $\{f_n\}$ be a uniformly pointwise convergent sequence of continuous functions on $X$. In particular, it is pointwise convergent at every $x \in X$, so it has a pointwise limit $f(x) = \lim_n f(x)$ for each $x$. We claim that $f(x)$ is continuous. Given $\varepsilon > 0$, choose $n_0$ sufficiently large so that for $m, n \geq n_0$ and for all $x \in X$ we have $|f_m(x) - f_n(x)| < \varepsilon$. This implies that $|f_n(x) - f(x)| \leq \varepsilon$ for all $x \in X$ and $n \geq n_0$. Fix $x_0 \in X$. Let $\delta > 0$ be such that for $d(x_0, y) < \delta$ we have $|f_n(x_0) - f_n(y)| < \varepsilon$. Then

$|f(x_0) - f(y_0)| \leq |f(x_0) - f_n(x_0)| + |f_n(x_0) - f_n(y)| + |f_n(y) - f(y)| \leq \varepsilon + |f_n(x_0) - f_n(y)| + \varepsilon < \varepsilon + \varepsilon + \varepsilon$

proving continuity. ///

[01.6] Show that $C^0[a, b]$ is not complete with the $L^2[a, b]$ metric.

Discussion: That is, we want a sequence $\{f_n\}$ of $C^0$ functions that is Cauchy in the $L^2$ metric, but not in the $C^0$ metric. In particular, it would suffice to find $\{f_n\}$ which converge in $L^2$ to an $L^2$ function which is not $C^0$.  

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For example, \( \{f_n\} \) can be a sequence of continuous, piecewise-linear functions converging pointwise to a step function (which is certainly not continuous). For example, with \([a, b] = [0, 1]\),

\[
f_n(x) = \begin{cases} 
0 & \text{(for } 0 \leq x < \frac{1}{2} - \frac{1}{n} \text{)} \\
\frac{n}{2} \cdot (x - \frac{1}{2} + \frac{1}{n}) & \text{(for } \frac{1}{2} - \frac{1}{n} \leq x \leq \frac{1}{2} + \frac{1}{n} \text{)} \\
1 & \text{(for } \frac{1}{2} + \frac{1}{n} < x \leq 1 \text{)}
\end{cases}
\]

The graph is flat to the left and flat to the right, and has a straight line of slope \( n/2 \) connecting the two flat parts. The pointwise limit is a step function with step of height 1 at \( \frac{1}{2} \).

For \( m \leq n \) the \( L^2 \) norm of \( f_m - f_n \) is easily estimated by

\[
|f_m - f_n|^2_{L^2} = \int_{\frac{1}{2} - \frac{1}{n}}^{\frac{1}{2} + \frac{1}{n}} |f_m(x) - f_n(x)|^2 \, dx \leq \int_{\frac{1}{2} - \frac{1}{n}}^{\frac{1}{2} + \frac{1}{n}} 1 \, dx \leq \frac{2}{m}
\]

Thus, the sequence is \( L^2 \)-Cauchy. Since the limit is not continuous, the sequence cannot possibly be \( C^0 \)-Cauchy. Explicitly, \( |f_m - f_n|_{C^0} = 1 \) for \( m \neq n \). ///

**[01.7]** Show that \( C^1[a, b] \) is not complete with the \( C^0[a, b] \) metric.

**Discussion:** One approach is to find a \( C^0 \)-Cauchy sequence of \( C^1 \) functions whose limit is not \( C^1 \). For example, in words, a tent function with base \([a, b] \) with vertex at the point \((\frac{2+k}{2}, 1)\) is continuous, but not differentiable. It can be approximate in \( C^0 \) by tent functions that are smoothed off in tinier-and-tinier intervals around the vertex.

Formulically, it’s a question of writing formulas for (for example) little pieces of pointer-and-pointer parabola pieces to replace the sharp corner at the peak of the tent function.

Losing interest in this approach... Is there a better one? Non-formulica? Seriously, turning obvious pictures into formulas quickly becomes unrewarding and non-explanatory...

Yes: we should soon prove that \( C^\infty[a, b] \) is dense in all the spaces \( C^k[a, b] \). This changes the presentation of the question, but annihilates it. ///

**[01.8]** Show that \( C^1[a, b] \) is complete, with the \( C^1[a, b] \) metric

\[
d(f, g) = \sup_{a \leq x \leq b} |f(x) - g(x)| + \sup_{a \leq x \leq b} |f'(x) - g'(x)|
\]

**Discussion:** For a Cauchy sequence \( \{f_i\} \) in \( C^k[a, b] \), the pointwise limits \( \lim f(x) \) and \( \lim f'(x) \) exist, and are continuous, since the limits are uniform pointwise. The issue is to show that \( \lim f \) is differentiable, with derivative \( \lim f' \). That is, for a Cauchy sequence \( f_n \) in \( C^1[a, b] \), with pointwise limits \( f(x) = \lim_n f_n(x) \) and \( g(x) = \lim_n f'_n(x) \), we have \( g = f' \). By the fundamental theorem of calculus, for any index \( i \),

\[
f_i(x) - f_i(a) = \int_a^x f'_i(t) \, dt
\]

Since the \( f'_i \) uniformly approach \( g \), given \( \varepsilon > 0 \) there is \( i_\varepsilon \) such that \( |f'_i(t) - g(t)| < \varepsilon \) for \( i \geq i_\varepsilon \) and for all \( t \) in the interval, so for such \( i \)

\[
\left| \int_a^x f'_i(t) \, dt - \int_a^x g(t) \, dt \right| \leq \int_a^x |f'_i(t) - g(t)| \, dt \leq \varepsilon \cdot |x - a| \to 0
\]
Thus,\[
\lim_{i} f_{i}(x) - f_{i}(a) = \lim_{i} \int_{a}^{x} f'_{i}(t) \, dt = \int_{a}^{x} g(t) \, dt
\]
from which \( f' = g \). ///

[01.9] Show that the Hilbert cube
\[
C = \{(z_{1}, z_{2}, \ldots) \in \ell^{2} : |z_{n}| \leq \frac{1}{n}\}
\]
is compact. More generally, for any sequence of positive reals \( r_{n} \),
\[
C(r) = \{(z_{1}, z_{2}, \ldots) \in \ell^{2} : |z_{n}| \leq r_{n}\}
\]
is compact if and only if \( \sum_{n} |r_{n}|^{2} < \infty \).

Discussion: Use the total boundedness criterion. Given \( \varepsilon > 0 \), by convergence of \( \sum_{n} \delta_{n}^{2} \), there is \( n_{0} \) large enough so that \( \sum_{n \geq n_{0}} \delta_{n}^{2} < \varepsilon^{2} \). The set
\[
C_{n_{0}} = \{(z_{1}, z_{2}, \ldots, z_{n_{0}}) \in \mathbb{R}^{n_{0}} : |z_{n}| \leq \delta_{n}\}
\]
is a compact subset of \( \mathbb{R}^{n_{0}} \), so certainly has a finite cover by open balls of radius \( \varepsilon \). Let the centers of these balls be \( w_{1}, \ldots, w_{N} \). Let \( j : \mathbb{R}^{n_{0}} \to \ell^{2} \) be the inclusion \( j(z_{1}, \ldots, z_{n_{0}}) = (z_{1}, \ldots, z_{n_{0}}, 0, 0, \ldots) \). Then we claim that the open balls of radius \( 2\varepsilon \) at \( j(w_{1}), j(w_{2}), \ldots, j(w_{N}) \) cover \( C(\delta) \). Indeed, given \( z = (z_{1}, z_{2}, \ldots) \in C(\delta) \), write \( z = j(z') + z'' \) where \( z' = (z_{1}, \ldots, z_{n_{0}}) \) and \( z'' = z - j(z') = (0, \ldots, 0, z_{n_{0}+1}, \ldots) \). There is at least one of the \( w_{j} \)'s within \( \varepsilon \) of \( z' \); let \( w_{j_{0}} \) be such. By the triangle inequality for the norm \( | \cdot |_{\ell^{2}} \) on \( \ell^{2} \),
\[
d(z, j(w_{j_{0}})) = |z - j(w_{j_{0}})|_{\ell^{2}} = |j(z') + z'' - j(w_{j_{0}})|_{\ell^{2}} \leq |j(z') - j(w_{j_{0}})|_{\ell^{2}} + |z''|_{\ell^{2}} = |z' - w_{j_{0}}| + |z''|_{\ell^{2}} < \varepsilon + \varepsilon
\]
Thus, \( C(r) \) can be covered by finitely-many open balls of radius \( 2\varepsilon \).

Conversely, if \( \sum_{n} r_{n}^{2} = +\infty \), then there are indices \( 1 \leq n_{1} < n_{2} < \ldots \) such that
\[
\sum_{n_{k} < i \leq n_{k+1}} r_{i}^{2} \geq 1
\]
With standard basis \( \{e_{n}\} \), let\[
v_{k} = \sum_{n_{k} < i \leq n_{k+1}} r_{i} \cdot e_{i}
\]
Then for \( k \neq \ell \),\[
|v_{k} - v_{\ell}|^{2} = \sum_{n_{k} < i \leq n_{k+1}} r_{i}^{2} + \sum_{n_{\ell} < i \leq n_{\ell+1}} r_{i}^{2} \geq 1 + 1
\]
Thus, there are no convergent subsequences, and \( C(r) \) is not sequentially compact, so not compact. ///

[01.10] Let \( | \cdot |_{1} \) and \( | \cdot |_{2} \) be two norms on a real or complex vector space \( X \). Suppose that \( |x|_{1} \geq |x|_{2} \) for all \( x \in X \). Let \( X_{1} \) be the completion of \( X \) with respect to the metric associated to \( | \cdot |_{1} \). Show that the identity map \( X \to X \) extends by continuity to a continuous injection \( X_{1} \to X_{2} \).

Discussion: As usual, attempt to define the extension-by-continuity \( S \) of the identity map by \( S(X_{1} - \lim x_{n}) = X_{2} - \lim x_{n} \) for \( x_{n} \in X \). Then we’d want or need to show that it is well-defined, that it is continuous, and linear, and that it is injective. All but the injectivity are treated in excruciating detail in the notes.

For injectivity, it is probably best to not attempt to prove this directly by purely elementary means. It is a significant issue, though, so we’ll come back to this later.