Examples discussion 02

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[This document is http://www.math.umn.edu/~garrett/m/real/examples_2018-19/real-disc-02.pdf]

[02.1] The space of continuous functions on $\mathbb{R}$ going to 0 at infinity is

$$C^o_0(\mathbb{R}) = \{ f \in C^o(\mathbb{R}) : \text{for every } \varepsilon > 0 \text{ there is } T \text{ such that } |f(x)| < \varepsilon \text{ for all } |x| \geq T \}$$

Show that the closure of $C^o_c(\mathbb{R})$ in the space $C^o_{\text{bdd}}(\mathbb{R})$ of bounded continuous functions with sup norm, is $C^o_0(\mathbb{R})$.

Discussion: The argument for this is general enough that we can replace $\mathbb{R}$ by a more general topological space $X$, probably locally compact and Hausdorff so that Urysohn’s lemma assures us a good supply of continuous functions for auxiliary purposes. Then $C^o_0(X)$ is defined to be the collection of continuous functions $f$ such that, given $\varepsilon > 0$, there is a compact $K \subset X$ such that $|f(x)| < \varepsilon$ for $x \not\in K$.

First, show that any $f \in C^o_0(\mathbb{R})$ is a sup-norm limit of functions from $C^o_0(\mathbb{R})$. Given $\varepsilon > 0$, let $K$ be sufficiently large so that $|f(x)| < \varepsilon$ for $x \not\in K$. We claim that there is an open $U \supset K$ with compact closure $\overline{U}$ (which would be obvious on $\mathbb{R}$ or $\mathbb{R}^n$). For each $x \in K$, let $U_x \ni x$ be an open set with compact closure (using the local compactness). By compactness of $K$, there is a finite subcover $K \subset U_{x_1} \cup \ldots \cup U_{x_n}$. Then the closure of $U = U_{x_1} \cup \ldots \cup U_{x_n}$ is compact, as claimed. Then, invoking Urysohn’s Lemma, let $\varphi$ be a continuous function on $X$ taking values in the interval $[0,1]$, that is 1 on $K$, and 0 off $U$, so $\varphi$ has compact support. Then $\varphi \cdot f$ is continuous and has compact support, and

$$\sup_{x \in X} |f(x) - \varphi(x) \cdot f(x)| \leq \sup_{x \in K} |f(x) - \varphi(x) \cdot f(x)| + \sup_{x \not\in K} |f(x) - \varphi(x) \cdot f(x)| = 0 + \sup_{x \not\in K} |f(x) - \varphi(x) \cdot f(x)|$$

$$\leq \sup_{x \not\in K} |1 - \varphi| \cdot \sup_{x \not\in K} |f(x)| < 1 \cdot \varepsilon$$

That is, we can approximate $f$ to within $\varepsilon$, as claimed.

On the other hand, now show that any sup-norm Cauchy sequence of $f_n \in C^o_0(X)$ has a pointwise limit $f$ in $C^o_0(X)$. First, on any compact, the limit of the $f_n$’s is uniform pointwise, so is continuous on compacts. Since every point $x \in X$ has a neighborhood $U_x$ with compact closure, the pointwise limit is continuous on $U_x$. Thus, the pointwise limit is continuous at every point, hence continuous. Given $\varepsilon > 0$, take $n_0$ sufficiently large so that $\sup_{x \in X} |f_m(x) - f_n(x)| < \varepsilon$ for all $m,n \geq n_0$. Let $K$ be the support of $f_{n_0}$. Then

$$\sup_{x \not\in K} |f(x)| = \sup_{x \not\in K} |f(x) - f_{n_0}(x)| \leq \sup_{x \in X} |f(x) - f_{n_0}| \leq \varepsilon$$

Thus, the pointwise limit goes to 0 at infinity. ///

[02.2] Show that $|\int_a^b f|^2 \leq |b-a| \cdot \int_a^b |f|^2$.

Discussion: This is the Cauchy-Schwarz-Bunyakowsky inequality on $L^2[a,b]$, where the inner product is

$$\langle f, g \rangle = \int_a^b f g = \int_a^b f(x) \overline{g(x)} \, dx$$

$$|\int_a^b f|^2 = \left| \int_a^b 1 \cdot f(x) \, dx \right|^2 \leq \int_a^b 1 \cdot \int_a^b |f|^2 = |b-a| \cdot \int_a^b |f|^2$$

[02.3] In $\ell^2$, show that the unique point in the closed unit ball closest to a point $v$ not inside that ball is $v/|v|_{\ell^2}$. 

1
Thus, the sum of the two inner products is positive, and by Cauchy-Schwarz-Bunyakowsky:

$$2|v| < \langle v, w \rangle + \langle w, v \rangle$$

Thus, 1 < |w|, which is impossible.

\\[02.4\\]

One form of the sawtooth function is $f(x) = x - \pi$ on $[0, 2\pi]$. Compute the Fourier coefficients $\hat{f}(n)$. From Plancherel-Parseval’s theorem for this function, show that

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \ldots = \frac{\pi^2}{6}$$

Discussion: We have the orthonormal basis $e_n(x) = \frac{1}{\sqrt{2\pi}}e^{inx}$ with $n \in \mathbb{Z}$ for the Hilbert space $L^2[0, 2\pi]$.

The Fourier coefficients are determined by Fourier’s formula

$$\hat{f}(n) = \int_0^{2\pi} f(x) \frac{e^{-inx}}{\sqrt{2\pi}} \, dx$$

For $n = 0$, this is 0. For $n \neq 0$, integrate by parts, to get

$$\hat{f}(n) = \left[ f(x) \cdot \frac{e^{-inx}}{\sqrt{2\pi} \cdot (-in)} \right]_0^{2\pi} - \int_0^{2\pi} \frac{1}{\sqrt{2\pi} \cdot (-in)} \cdot -\frac{e^{-inx}}{(-in)} \, dx$$

$$= \left( \pi \cdot \frac{1}{\sqrt{2\pi} \cdot (-in)} \right) - \left( -\pi \cdot \frac{1}{\sqrt{2\pi} \cdot (-in)} \right) = \frac{2\pi}{\sqrt{2\pi} \cdot (-in)} = \frac{\sqrt{2\pi}}{-in}$$

The $L^2$ norm of $f$ is

$$\int_0^{2\pi} (x - \pi)^2 \, dx = \left[ \frac{(x - \pi)^3}{3} \right]_0^{2\pi} = \frac{\pi^3 - (-\pi)^3}{3} = \frac{2\pi^3}{3}$$

Thus, by Parseval,

$$\sum_{n \neq 0} \left| \frac{\sqrt{2\pi}}{-in} \right|^2 = \frac{2\pi^3}{3}$$
Paul Garrett: Examples discussion 02 (October 28, 2018)

This simplifies first to
\[
2 \sum_{n \geq 1} \frac{2\pi}{n^2} = \frac{2\pi^3}{3}
\]
and then to
\[
\sum_{n \geq 1} \frac{1}{n^2} = \frac{\pi^2}{6}
\]
That is, Parseval applied to the sawtooth function evaluates \(\zeta(2)\).

[02.5] Show that there is no \(f_o \in C^\infty[0,1]\) so that, for all \(g \in C^\infty[0,1]\), \(\int_0^1 f_o(x) g(x) \, dx = g\left(\frac{1}{2}\right)\).

**Discussion:** Here is just one among many possible approaches. By Cauchy-Schwarz-Bunyakowsky in \(L^2[0,1]\) with its usual inner product, for every \(g \in C^\infty[0,1]\) we’d have
\[
|g(\frac{1}{2})| = \left| \int_0^1 f_o(x) g(x) \, dx \right| = |g, f_o| \leq |g|_{L^2} \cdot |f_o|_{L^2}
\]
That is, supposedly \(g\left(\frac{1}{2}\right)\) would be bounded by a constant multiple of \(|g|_{L^2}\), for every \(g \in C^\infty\). But this is not true: we can make a variety of sequences \(\{g_n\}\) of continuous functions with support in \([\frac{1}{2} - \frac{1}{n}, \frac{1}{2} + \frac{1}{n}]\), with \(g_n\left(\frac{1}{2}\right) = 1\), and with sup \(|g_n| = 1\). Piecewise-linear tent functions of height 1 and base \([\frac{1}{2} - \frac{1}{n}, \frac{1}{2} + \frac{1}{n}]\) would do. The \(L^2\) norms go to 0 as \(n \to +\infty\).

[02.6] For \(c_1 > c_2 > c_3 > \ldots > 0\) a monotone-decreasing sequence of positive reals, with \(\lim_n c_n = 0\), show that, for every \(0 < x < 2\pi\), \(\sum_n c_n e^{inx}\) converges.

**Discussion:** The expression as a Fourier series should not distract us from seeing an instance of the generalized alternating-decreasing criterion again, sometimes called Dirichlet’s criterion; for a positive real sequence \(c_1, c_2, \ldots\) monotone-decreasing to 0, and for a (possibly complex) sequence \(b_1, b_2, \ldots\) with bounded partial sums \(B_n = b_1 + \ldots + b_n\), the sum \(\sum_n b_n c_n\) converges. The partial sums \(\sum_{n \leq N} e^{2\pi inx}\) are bounded for \(0 < x < 1\), by summing finite geometric series:
\[
\left| \sum_{n=-M}^{N} z^n \right| = \frac{|z^{-M} - z^{N+1}|}{|1 - z|} \leq \frac{2}{|1 - z|}
\]
so this criterion applies here.

The proof of the criterion itself is by summation by parts, a discrete analogue of integration by parts. That is, rewrite the tails of the sum as
\[
\sum_{M \leq n \leq N} b_n c_n = \sum_{M \leq n \leq N} (B_n - B_{n-1}) c_n = -B_{M-1} c_M + \sum_{M \leq n \leq N} B_n (c_n - c_{n+1}) + B_N c_{N+1}
\]
Since the partial sums are bounded, the first and last summand go to 0. Letting \(\beta\) be a bound for all the \(|B_n|\), the summation is
\[
\left| \sum_{M \leq n \leq N} B_n (c_n - c_{n+1}) \right| \leq \sum_{M \leq n \leq N} |B_n| |c_n - c_{n+1}| = \sum_{M \leq n \leq N} |B_n| (c_n - c_{n+1}) \leq \sum_{M \leq n \leq N} \beta \cdot (c_n - c_{n+1})
\]
by telescoping the series. Again, \(c_M\) and \(c_{N+1}\) go to 0. //
[02.7] Let \( b = \{ b_n \} \) be a sequence of complex numbers, such that there is a bound \( B \) such that, for every \( c = \{ c_n \} \in \ell^2, \sum_n b_n c_n \leq B \cdot |c|_{\ell^2} \). Show that \( b \in \ell^2 \).

**Discussion:** The assumed inequality says that \( \lambda(c) = \sum_n b_n c_n \) is a bounded linear functional on \( \ell^2 \). By Riesz-Fréchet, there is \( a = (a_1, a_2, \ldots) \in \ell^2 \) such that \( \lambda(c) = \sum_n a_n c_n \) for all \( c \in \ell^2 \). Then, with \( \{ e_n \} \) the standard (Hilbert-space) basis for \( \ell^2 \), \( b_n = \lambda(e_n) = a_n \) proves that \( a = b \), so \( b \in \ell^2 \).

[02.8] For a vector subspace \( W \) of a Hilbert space \( V \), show that \((W^\perp)^\perp \) is the topological closure of \( W \).

**Discussion:** Let \( \lambda_x(v) = \langle v, x \rangle \) for \( x, v \in V \). Then \( W^\perp = \bigcap_{w \in W} \ker \lambda_w \). Similarly, \( (W^\perp)^\perp = \bigcap_{x \in W^\perp} \ker \lambda_x \).

From the discussion in the Riesz-Fréchet theorem, or directly via Cauchy-Schwarz-Bunyakowsky, \( \lambda_x \) is continuous, so \( \ker \lambda_x = \lambda_x^{-1}(\{0\}) \) is closed, since \( \{0\} \) is closed. (One might check that the kernel of a linear map is a vector subspace.) An arbitrary intersection of closed sets is closed, so \((W^\perp)^\perp\) is closed.

Certainly \((W^\perp)^\perp \supset W \), because for each \( w \in W \), \( \langle x, w \rangle = 0 \) for all \( x \in W^\perp \). Thus, \((W^\perp)^\perp \) is a closed subspace, containing \( W \). Being a closed subspace of a Hilbert space, \((W^\perp)^\perp \) is a Hilbert space itself. If \((W^\perp)^\perp\) were strictly larger than the topological closure \( W^\perp \) of \( W \), then there would be \( 0 \neq y \in (W^\perp)^\perp \) orthogonal to \( W^\perp \) itself, contradicting \( 0 \neq y \in (W^\perp)^\perp \).

[02.9] Find two dense vector subspaces \( X, Y \) of \( \ell^2 \) such that \( X \cap Y = \{ 0 \} \). (And, if you need further entertainment, can you find countably-many dense vector subspaces \( X_n \) such that \( X_m \cap X_n = \{ 0 \} \) for \( m \neq n \)?)

**Discussion:** First, as a variant that refers to more natural constructions, but requires non-trivial proofs to fully validate it, we can make two dense subspaces of \( L^2[0,1] \) which intersect just at \( \{0\} \). Namely, the vector space of all finite Fourier series, and the vector space of all polynomials (restricted to \( [0,1] \)). We need to know that the appropriate exponentials (or sines and cosines) give a Hilbert space basis of \( L^2[0,1] \), and also Weierstraß' result on the density of polynomials in \( C^n[0,1] \), hence (depending on our definitional set-up) in \( L^2[0,1] \).

A more elementary, but trickier, approach is the following. Let \( X \) be the vector space of finite linear combinations of the standard Hilbert space basis \( \{ e_n \} \). This is a natural subspace. For the other subspace \( Y \), some sort of trickery seems to be needed, either in specification of \( Y \) itself so as to make verification of \( X \cap Y = \{ 0 \} \) easy, or a simpler specification of \( Y \) but with complicated verification that \( X \cap Y = \{ 0 \} \), or both.

One possibility involves Sun-Ze's theorem (sometimes called the Chinese Remainder Theorem), namely, that for a finite collection of mutually relatively prime integers \( N_1, \ldots, N_k \), and for integers \( b_1, \ldots, b_k \) there exists \( x \in \mathbb{Z} \) such that \( x = b_k \mod N_k \). Further, this \( x \) can be arbitrarily large, by adding multiples of the product \( N_1 \cdots N_k \) to it. Let \( p_n \) be the \( n \)th prime number, and put

\[
v_n = e_n + \sum_{k \geq 1} \frac{1}{kp_n} \cdot e_{kp_n}
\]

Of course, we claim that no (non-zero) finite linear combination \( y = \sum_n c_n \cdot v_n \) is in \( X \). That is, we claim that for any such non-zero linear combination, there are arbitrarily large indices \( \ell \) such that \( \langle y, e_{\ell} \rangle \neq 0 \). Let \( n_o \) be the largest index \( n \) such that \( e_n \neq 0 \). Invoking Sun-Ze's theorem, there exist \( \ell \geq n_o \) such that \( \ell = 1 \mod p_i \) for \( i < n_o \) and \( \ell = 0 \mod p_{n_o} \). Then

\[
\langle y, e_\ell \rangle = \sum_{n} \left( \frac{1}{n} \langle e_n, e_\ell \rangle + \sum_{k} \frac{1}{kp_n} \langle e_{kp_n}, e_\ell \rangle \right) = \sum_{n < n_o} 0 + \frac{1}{\ell} \not= 0
\]

This proves that \( X \cap Y = \{ 0 \} \).
Certainly $X$ is dense, because every vector in $\ell^2$ is an infinite sum of vectors from $X$, that is, an $\ell^2$ limit of finite linear combinations of vectors from $X$.

To see that $Y$ is dense, observe that applying an infinite version of Gram-Schmidt to the vectors $v_n$ produces the standard basis $e_n$. That is, the $e_n$’s are infinite linear combinations of the $v_n$’s, so $Y$ is dense. (Yes, there is an issue about convergence in an infinite version of Gram-Schmidt, in general!)