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Review examples 00

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[This document is http://www.math.umn.edu/~garrett/m/real/examples_2018-19/real-ex-00.pdf]

If you want feedback on your write-ups on any of these examples, please get me your write-ups by Friday, 14 Sept, 2018.

[00.1] (There is not much hope to make sense of the outcome of an uncountable number of non-zero operations:) Let $\Omega$ be an uncountable collection of positive real numbers. Letting $F$ range over all finite subsets of $\Omega$, show that $\sup_F \sum_{\alpha \in F} \alpha = +\infty$.

[00.2] Prove (or review the proof) that a continuous real-valued function $f$ on a finite closed interval $[a, b] \subset \mathbb{R}$ is uniformly continuous: for all $\varepsilon > 0$ there is $\delta > 0$ such that, for all $x, y \in [a, b]$, $|x - y| < \delta$ implies $|f(x) - f(y)| < \varepsilon$.

[00.3] Prove (or review the proof) that a uniform pointwise limit of continuous, real-valued functions on $[a, b]$ is continuous.

[00.4] Prove (or review the proof) of the Fundamental Theorem of Calculus: for a continuous function $f$ on $[a, b]$, the function $F(x) = \int_a^x f(t) dt$ is continuously differentiable, and has derivative $f$. (Use Riemann’s integral.)

[00.5] Prove (or review the proof) that for a sequence of real-valued functions $f_n$ on $[0, 1]$ approaching $f$ uniformly pointwise, $\lim_n \int_0^1 f_n(x) dx = \int_0^1 \lim_n f_n(x) dx$. (Use Riemann’s integral.)

[00.6] Show that every open subset of $\mathbb{R}$ is a countable union of open intervals.

[00.7] Define Lebesgue (outer) measure $\mu(E)$ of subsets $E$ of $\mathbb{R}$ by

$$\mu(E) = \inf \left\{ \sum_{n=1}^{\infty} |b_n - a_n| : E \subset \bigcup_{n=1}^{\infty} (a_n, b_n) \right\}$$

Show that $\mu(\mathbb{Q}) = 0$. Show that $\mu(M) = 0$, where $M$ is Cantor’s middle-thirds set.