Examples 04

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[This document is http://www.math.umn.edu/~garrett/m/real/examples_2018-19/real-ex-04.pdf]

For feedback on these examples, please get your write-ups to me by Friday, February 08, 2019.

[04.1] With \( g(x) = f(x + x_o) \), express \( \hat{g} \) in terms of \( \hat{f} \), for \( f \in L^1(\mathbb{R}^n) \).

[04.2] Let \( \{b_n\} \) be a sequence of complex numbers. Suppose that, for every \( \{a_n\} \in \ell^2 \), \( \sum_n a_n b_n \) converges. Show that \( \{b_n\} \in \ell^2 \).

[04.3] Let \( g \) be a measurable \([0, +\infty]\)-value function on \([a, b]\) such that, for every \( f \in L^2[a, b] \), \( \int_a^b \left| f(x) g(x) \right| \, dx < \infty \). Show that \( g \in L^2[a, b] \).

[04.4] Give a persuasive proof that the function

\[
  f(x) = \begin{cases} 
  0 & \text{(for } x \leq 0) \\
  e^{-1/x} & \text{(for } x > 0) 
  \end{cases}
\]

is infinitely differentiable at 0. Use this to make a smooth step function: 0 for \( x \leq 0 \) and 1 for \( x \geq 1 \), and goes monotonically from 0 to 1 in the interval \([0, 1]\). Use this to construct a family of smooth cut-off functions \( \{f_n : n = 1, 2, 3, \ldots\} \): for each \( n \), \( f_n(x) = 1 \) for \( x \in [-n, n] \), \( f_n(x) = 0 \) for \( x \notin [-(n+1), n+1] \), and \( f_n \) goes monotonically from 0 to 1 in \([-n, n]\) and monotonically from 1 to 0 in \([n, n+1]\).

[04.5] Give an explicit non-zero function \( f \) such that \( \int_\mathbb{R} x^n f(x) \, dx = 0 \).

[04.6] Show that \( \chi_{[a,b]} \ast \chi_{[c,d]} \) is a piecewise-linear function, and express it explicitly.

[04.7] Compute \( e^{-\pi x^2} \ast e^{-\pi x^2} \) and \( \frac{\sin x}{x} \ast \frac{\sin x}{x} \). (Be careful what you say: \( \frac{\sin x}{x} \) is not in \( L^1(\mathbb{R}) \), so there are potential problems with convolution.)

[04.8] For \( f \in \mathcal{S} \), show that

\[
  \lim_{\varepsilon \to 0^+} f(x) \ast e^{-\pi x^2 / \varepsilon} = f(x)
\]

[04.9] For \( f \in \mathcal{S} \), show that

\[
  \lim_{t \to \infty} f(x) \ast \frac{2 \sin tx}{tx} = f(x)
\]

[04.10] Evaluate the Borwein integral

\[
  \int_\mathbb{R} \frac{\sin x}{x} \cdot \frac{\sin x/3}{x/3} \cdot \frac{\sin x/5}{x/5} \, dx
\]