\[06.1\] On \( T \), show that \( u'' = \delta_Z \) has no solution \( u \in \mathcal{D}^* \).

**Discussion:** Let \( \psi_n(x) = e^{2\pi inx} \). We can use Fourier expansions for every element of \( \mathcal{D}^* = \mathcal{D}(T)^* \), because by Sobolev imbedding \( \mathcal{D}^* = H^{-\infty} \). Also, because differentiation is a continuous map on \( H^{-\infty} \), Fourier series can always be differentiated termwise. Thus, the equation
\[
\left( \frac{d}{dx} \right)^2 \left( \sum_n \hat{u}(n)\psi_n \right) = u'' = \delta_Z = \sum_n 1 \cdot \psi_n
\]
is equivalent to coefficient-wise equality, which is \((2\pi in)^2 \hat{u}(n) = 1\). This is impossible for \( n = 0 \). ///

\[06.2\] Define *translation* \( T_{x_o}u \) of a distribution \( u \) by an amount \( x_o \in \mathbb{R} \) by
\[
(T_{x_o}u)(\varphi) = u(T_{-x_o}\varphi) \quad \text{for} \ \varphi \in \mathcal{D}
\]
The sign is for compatibility with distributions arising as integrate-against test functions. For tempered \( u \), express \( T_{x_o}u \) in terms of \( \hat{u} \).

**Discussion:** Given the compatibility with ordinary Fourier transform on nice functions, taking into account the integration-against aspect, it suffices to determine the relation on those nice functions: by changing variables, replacing \( x \) by \( x - x_o \),
\[
f(x + x_o)\hat{\gamma}(\xi) = \int_{\mathbb{R}} e^{-2\pi i \xi x} f(x + x_o) \, dx = \int_{\mathbb{R}} e^{-2\pi i \xi (x - x_o)} f(x) \, dx = e^{2\pi i \xi x_o} \cdot \hat{f}(\xi)
\]
Because we want distributions to extend integrating-against-functions, and since
\[
\int_{\mathbb{R}} T_{x_o}f(x) F(x) \, dx = \int_{\mathbb{R}} f(x + x_o) F(x) \, dx = \int_{\mathbb{R}} f(x) F(x - x_o) \, dx
\]
so the correct definition of \( T_{x_o}u \) is
\[
(T_{x_o}u)(\varphi) = u(T_{-x_o}\varphi)
\]
Thus, in these conventions, there is a sign flip: for tempered distributions \( u \),
\[
\widehat{T_{x_o}u} = e^{-2\pi i \xi x_o} \cdot \hat{u}
\]
This confusion about signs seems to be inescapable. ///

\[06.3\] Compute \( \hat{\cos x} \).

**Discussion:** Using the previous example, letting \( T_{x_o} \) be translation by \( x_o \),
\[
2 \cos x = e^{2\pi ix} \cdot 1 + e^{-2\pi ix} \cdot 1 = e^{2\pi ix} \cdot \hat{\delta} + e^{-2\pi ix} \cdot \hat{\delta} = \widehat{T_{x_o} \delta} + \widehat{T_{-x_o} \delta}
\]
By Fourier inversion (on tempered distributions, and using the fact that cosine is *even*),
\[
\hat{\cos x} = \frac{1}{2} \left( T_{x_o} \delta + T_{-x_o} \delta \right) = \frac{1}{2} (\delta(\xi - x) + \delta(\xi + x))
\]
where the latter expresses the intent/idea of the thing, though is slightly imprecise. Note also the sign flips, which happen to have no impact on the outcome, since cosine is an even function.

[06.4] On $\mathbb{R}^n$, show that $|x|^2 \cdot \Delta \delta = 2n \cdot \delta$.

Discussion: First, for $\varphi \in \mathcal{D}$,

$$
(|x|^2 \cdot \Delta \delta)(\varphi) = \delta(\Delta(|x|^2 \cdot \varphi))
$$

By direct computation,

$$
\Delta(|x|^2 \cdot \varphi) = \sum_j \left( 2 \cdot \varphi + 4x_j \frac{\partial \varphi}{\partial x_j} + r^2 \frac{\partial^2 \varphi}{\partial x_j^2} \right)
$$

Upon application of $\delta$, that is, evaluation at 0, all the terms vanish except the $2 \cdot \varphi$, which is summed from 1 to $n$, giving $2n \cdot \varphi(0) = 2n \cdot \delta(\varphi)$.

[06.5] Compute the Fourier transform of the sign function

$$
\text{sgn}(x) = \begin{cases} 
-1 & (x < 0) \\
+1 & (x > 0) 
\end{cases}
$$

Discussion: The sign function is odd, and of positive-homogeneous degree 0. Thus, by computations about the interaction of Fourier transform and Euler operator, its Fourier transform is also odd, and of degree $-(1 - 0)$, where the 1 is the dimension, and the 0 is the degree of the sign function.

We have seen that the principal value integral against $1/x$ is odd and of degree $-1$. The uniqueness theorem for homogeneous distributions of a given parity implies that the Fourier transform of the sign function must be a constant multiple of the principal value integral against $1/x$.

To determine the constant, apply both to an odd Schwartz function whose Fourier transform we understand, such as the iconic $xe^{-\pi x^2}$, whose Fourier transform is $-i$ times it. (Maybe later: determination of the constant is secondary.)

[06.6] Compute the two-dimensional Fourier transform of $(x \pm iy)^n \cdot e^{-\pi(x^2 + y^2)}$. (Hint: It is useful to rewrite things in terms of a complex variable $z = x + iy$ and its complex conjugate $\bar{z} = x - iy$.)

Discussion: Using the complex coordinates, Fourier transform is

$$
\hat{f}(w) = \int_\mathcal{C} e^{-2\pi i \text{Re}(z\bar{w})} f(z) \, dx \, dy = \int_\mathcal{C} e^{-\pi i (z\bar{w} + \bar{z}w)} f(z) \, dx \, dy
$$

Thus, with the plus sign in the $\pm$, since the Fourier transform of (a suitably normalized) Gaussian is itself,

$$
\int_\mathcal{C} e^{-\pi i (z\bar{w} + \bar{z}w)} z^n e^{-\pi z\bar{z}} \, dx \, dy = \frac{1}{(-\pi i)^n} \int_\mathcal{C} \left( \frac{\partial}{\partial \bar{w}} \right)^n e^{-\pi i (z\bar{w} + \bar{z}w)} e^{-\pi z\bar{z}} \, dx \, dy
$$

$$
= \frac{1}{(-\pi i)^n} \left( \frac{\partial}{\partial \bar{w}} \right)^n \int_\mathcal{C} e^{-\pi i (z\bar{w} + \bar{z}w)} e^{-\pi z\bar{z}} \, dx \, dy = \frac{1}{(-\pi i)^n} \left( \frac{\partial}{\partial \bar{w}} \right)^n e^{-\pi w\bar{w}}
$$

$$
= \frac{1}{(-\pi i)^n} (-\pi w)^n \cdot e^{-\pi w\bar{w}} = i^{-n} \cdot w^n e^{-\pi w\bar{w}}
$$
as claimed. Yes, one might take a moment to check that the usual symbol manipulation does extend correctly to this complex-variables variation.

\[06.7\] The Cauchy-Riemann operator on \( C \approx \mathbb{R}^2 \) is

\[
\bar{\partial} = \frac{\partial}{\partial z} = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right)
\]

Let \( u_{n,s}(z) = \left( \frac{z}{n!} \right)^n |z|^s \) for \( n \in \mathbb{Z} \) and \( s \in \mathbb{C} \). Determine the requirements on \( n, s \) such that \( u_{n,s} \) is locally integrable (and, thus, because it is of moderate growth, gives a tempered distributions). Compute \( \bar{\partial} u_{n,s} \), and explain how to interpret the outcome in case the outcome is no longer locally integrable.

**Discussion:** The local integrability for \( \text{Re}(s) > -2 \) is immediate, upon changing to polar coordinates. Just in symbols (which, with well-chosen symbols, surely suggest the correct outcomes),

\[
\bar{\partial} u_{n,s} = \frac{\partial}{\partial z} \left( \frac{z}{n!} \right)^n \left( \frac{z}{n!} \right)^{-n+s} = \frac{-n + s}{2} \cdot \left( \frac{z}{n!} \right)^{n+1} \cdot |z|^{s-1}
\]

As with \( |x|^s \) on \( \mathbb{R}^n \), we can regularize these distributions outside the range of local integrability by examining their behavior under \( \Delta = 4 \partial \bar{\partial} \). Namely,

\[
\Delta u_{n,s} = (s^2 - n^2) \cdot u_{n,s-2}
\]

Replacing \( s \) by \( s + 2 \) and rearranging,

\[
u_{n,s} = \frac{\Delta u_{n,s+2}}{s^2 - n^2}
\]

Thus, while we have local integrability in \( \text{Re}(s) > -2 \), the right-hand side of the latter expression gives a (tempered) distribution for \( \text{Re}(s + 2) > -2 \), that is, for \( \text{Re}(s) > -4 \). Iterating this process unambiguously defines a distribution for all \( s \in \mathbb{C} \) away from \(-2, -4, -6, \ldots\).

\[06.8\] On \( \mathbb{R}^n \), for fixed \( \varphi \in D \), show that the function \( f_\varphi(s) = \int_{\mathbb{R}^n} \varphi(x) |x|^s \, dx \) blows up as \( s \to -n^+ \), in particular, there is a constant \( C_n \) such that

\[
f_\varphi(s) = \frac{C_n \cdot \varphi(0)}{s + n} \quad \text{(continuous at } -n)\]

(Thus, if we understand that \( s \to \) integration-against \( |x|^s \) is a meromorphic distribution-valued function, its residue at \( s = -n \) is a constant multiple of \( \delta \)).

**Discussion:** Let \( \psi(x) = e^{-\pi x^2} \). Then

\[
f_\varphi(s) = \int_{\mathbb{R}^n} \psi(x) |x|^s \, dx = \left| S^{n-1} \right| \cdot \int_0^\infty e^{-\pi r^2} r^s \, dr = \left| S^{n-1} \right| \cdot \frac{1}{2} \int_0^\infty e^{-\pi r} r^{\frac{s+n}{2}} \, dr
\]

\[
= \left| S^{n-1} \right| \cdot \frac{1}{2} \pi^{-\frac{s+n}{2}} \int_0^\infty e^{-r} r^{\frac{s+n}{2}} \, dr = \left| S^{n-1} \right| \cdot \frac{1}{2} \pi^{-\frac{s+n}{2}} \cdot \Gamma \left( \frac{s+n}{2} \right)
\]

The Gamma function \( \Gamma(z) \) has a simple pole at \( z = 0 \), so the latter expression blows up (in that sense) at \( s + n = 0 \), which is \( s = -n \). More precisely,

\[
\Gamma(s) = \frac{1}{s} \quad \text{(holomorphic at } s = 0)\]

so

\[
\Gamma \left( \frac{s+n}{2} \right) = \frac{2}{s+n} \quad \text{(holomorphic at } s = -n)\]
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and

\[ f_\psi(s) = |S^{n-1}| \cdot \frac{1}{2} \cdot \frac{2}{s + n} + (\text{holomorphic at } s = -n) = \frac{|S^{n-1}|}{s + n} + (\text{holomorphic at } s = -n) \]

Certainly \( \varphi \to f_\varphi(s) \) is \( \text{linear} \) in the argument \( \varphi \). Since \( F = \varphi - \varphi(0) \cdot \psi \) is 0 at 0, the integral for \( f_F(s) \) converges absolutely in \( \text{Re}(s) > -n - 1 \). Thus,

\[ f_\varphi(s) = f_F(s) + \varphi(0) \cdot f_\psi(s) = (\text{holo at } s = -n) + \varphi(0) \cdot \left( \frac{|S^{n-1}|}{s + n} + (\text{holo at } s = -n) \right) \]

\[ = \frac{\varphi(0) \cdot |S^{n-1}|}{s + n} + (\text{holo at } s = -n) \]

This holds for every \( \varphi \), so the residue at \( s = -n \) is \( |S^{n-1}| \) times \( \delta \).

\[ 06.9 \quad \text{The Riemann-equation characterizing holomorphic functions } f \text{ is } \partial f = 0. \text{ Show that} \]

\[ \partial \frac{1}{z} = (\text{constant multiple of}) \delta \]

**Discussion:** Yes, this fact mirrors the Cauchy formulas. Taking Fourier transform,

\[ -i\pi \mu \cdot \frac{1}{z} = 1 \]

The function/distribution \( 1/z \) is positive homogeneous of degree \( -1 \), and rotation-equivariant by \( \mu \to \mu^{-1} \).

Thus, its Fourier transform is homogeneous of degree \( -(2 - 1) = -1 \), and it has the same rotation equivariance. Applying \( \partial \) gives it rotation \( \text{invariance} \), and homogeneity degree \( (-1) - 1 = -2 \). By the uniqueness theorem, up to constants, this Fourier transform must be \( \delta \).

To determine the constant, apply both to a convenient test function. \///

\[ 06.10 \quad \text{On } \mathbb{R}^2 \approx \mathbb{C}, \text{ show that } Tf(z) = f(z)/z \text{ is a continuous map of the subspace } \mathcal{S}_1 = \{ f \in \mathcal{S}(\mathbb{R}^2) : f(\mu z) = \mu \cdot f(z) \forall |\mu| = 1 \} \text{ to } C^c(\mathbb{R}^2). \text{ (Hint: Use Taylor-Maclaurin series.)} \]

**Discussion:** [... iou ...]

\[ 06.11 \quad \text{On } \mathbb{R}^2 \approx \mathbb{C}, \text{ show that the principal-value integral} \]

\[ u(f) = \lim_{\varepsilon \to 0^+} \int_{|z| \geq \varepsilon} f(z) \frac{z}{|z|^3} \, dx \, dy \quad \text{(for } f \in \mathcal{S}) \]

gives a tempered distribution.

**Discussion:** As in other examples, and the one-dimensional principal-value integral against \( 1/x \), this integral is interesting because it is at the edge of the region of local integrability of functions \( z/|z|^s \), with \( s \in \mathbb{C} \). Understanding of it can be construed as an instance of \( \text{regularization} \).

Integrating by parts twice, using \( \Delta = 4\partial\overline{\partial} \), doing the requisite subordinate estimates, this functional is

\[ \lim_{\varepsilon \to 0^+} \int_{|z| \geq \varepsilon} f(z) \frac{z}{|z|^3} \, dx \, dy = -4 \lim_{\varepsilon \to 0^+} \int_{|z| \geq \varepsilon} \Delta f(z) \frac{z}{|z|^3} \, dx \, dy = -4 \int_\mathbb{C} \Delta f(z) \frac{z}{|z|^3} \, dx \, dy \]

since \( z/|z| \) is locally integrable. Thus, it gives a distribution. Also, it is (pointwise) \( \text{bounded} \), so certainly of suitably moderate growth to give a \( \text{tempered} \) distribution. \///
[06.12] Compute the Fourier transform of the distribution in the previous example.

Discussion: By either its direct definition or the equivalent regularized characterization from the previous example, this distribution is of homogeneous degree $-2$, and has rotation-equivariance (using the model $\mathbb{R}^2 = \mathbb{C}$) $\mu \to \mu^1$. From earlier discussion of behavior of rotation-equivariance and homogeneity under Fourier transform, up to a constant, its Fourier transform has the same rotational equivariance, and is of homogeneity degree $-(-2 - (-2)) = 0$. By the uniqueness theorem, this Fourier transform is a constant multiple of the (unique-up-to-constants) degree-zero distribution $z/|z|$ with the same rotation-equivariance.

To determine the constant, evaluate things at a Schwartz function with the same rotation-equivariance, and whose Fourier transform is understood, such as $z^n e^{-\pi z^2}$, for $n \geq 0$. ... [iou ...]