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Examples Discussion 08[draft]

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[This document is http://www.math.umn.edu/~garrett/m/real/notes_2019-20/real-disc-08.pdf]

[08.1] For abelian groups A, B, C, prove that $\mathrm{Hom}(A \otimes B, C) \approx \mathrm{Hom}(A, \mathrm{Hom}(B, C))$.

Discussion: This isomorphism (together with its nice properties) is the *adjunction* given by

$$\Phi \to \varphi_\Phi \quad \text{with} \quad \varphi_\Phi(a)(b) = \Phi(a \otimes b) \qquad \text{and} \qquad \Phi_\varphi \longleftarrow \varphi \quad \text{with} \quad \Phi_\varphi(a \otimes b) = \varphi(a)(b)$$

The map $\Phi_\varphi \longleftarrow \varphi$ uses properties of the tensor product. Specifically, $\beta_\varphi \longleftarrow \varphi$ with bilinear form $\beta_\varphi(a \times b) = \varphi(a)(b)$ makes immediate sense, and by properties of the tensor product $A \otimes B$ of abelian groups, a *bilinear* map $\beta_\varphi : A \times B \to C$ produces a unique *linear* map $\Phi_\varphi : A \otimes B \to C$ through which β_φ factors.

The latter remarks show that Φ_φ really is in $\mathrm{Hom}(A \otimes B, C)$. Similarly, φ_Φ really is in $\mathrm{Hom}(A, \mathrm{Hom}(B, C))$, that is, it is a linear map from A to $\mathrm{Hom}(B, C)$: indeed

$$\varphi_\Phi(a + a')(b) \;=\; \Phi((a + a') \otimes b) \;=\; \Phi(a \otimes b) + \Phi(a' \otimes b) \;=\; \varphi_\Phi(a)(b) + \varphi_\Phi(a')(b)$$

The maps are mutual inverses, at the level of sets: in one direction, for $\Psi \in \mathrm{Hom}(A \otimes B, C)$,

$$\Phi_{\varphi_\Psi}(a \otimes b) \;=\; \varphi_\Psi(a)(b) \;=\; \Psi(a \otimes b)$$

Similarly, in the other direction, for $\psi \in \mathrm{Hom}(A, \mathrm{Hom}(B, C))$,

$$\varphi_{\Phi_\psi}(a)(b) \;=\; \Phi_\psi(a \otimes b) \;=\; \psi(a)(b)$$

Additivity of $\Phi_\varphi \longleftarrow \varphi$: for $\psi \in \mathrm{Hom}(A, \mathrm{Hom}(B, C))$:

$$\Phi_{\varphi+\psi}(a \otimes b) \;=\; (\varphi + \psi)(a)(b) \;=\; \varphi(a)(b) + \psi(a)(b) \;=\; (\Phi_\varphi + \Phi_\psi)(a \otimes b)$$

Additivity in the opposite direction is similar. ///

[0.1] Remark: The *naturality* aspect of the adjunction, roughly, that the adjunction behaves well as A, B, C vary, does also require proof, but we are not using that aspect at the moment.

[08.2] For a field k and a k-vectorspace V (without topology), show that the map $V \otimes V^* \to \mathrm{End}_k V$ induced from $(v \otimes \lambda)(w) = \lambda(w) \cdot v$, for $v, w \in V$ and $\lambda \in V^*$, is a bijection to *finite-rank* endomorphisms of V (meaning that their images are finite-dimensional).

Discussion: Part of the point is that this is true over arbitrary fields k, and does not depend on having an inner product. Let $Tv \otimes \lambda(w) = \lambda(w) \cdot v$ for $v \in V$ and $\lambda \in V^*$.

First, to be sure that the map T exists and is unambiguous, we exactly must show that the map $V \times V^* \to \mathrm{End}_k V$ is *bilinear*, from which the existence and uniqueness of T will follow by the properties of tensor products. We carry out these routine verifications, to show how it's done. Additivity in the first argument:

$$((v + v') \times \lambda)(w) \;=\; \lambda(w) \cdot (v + v') \;=\; \lambda(w) \cdot v + \lambda(w) \cdot v'$$

and in the second argument:

$$(v \times (\lambda + \lambda'))(w) \;=\; (\lambda + \lambda')(w) \cdot v \;=\; \lambda(w) \cdot v + \lambda'(w) \cdot v$$

Similarly for scalars c:

$$(cv \times \lambda)(w) \;=\; \lambda(w) \cdot cv \;=\; c \cdot (\lambda(w) \cdot v) \;=\; c \cdot ((v \times \lambda)(w))$$

and

$$(v \times c\lambda)(w) \;=\; (c\lambda)(w) \cdot v \;=\; c \cdot (\lambda(w) \cdot v) \;=\; c \cdot ((v \times \lambda)(w))$$

So T exists, and is unambiguous.

Now, more interestingly, we show that $T(V \otimes V^*)$ consists of finite-rank endomorphisms of V. Indeed, every element of $V \otimes V^*$ is a *finite* sum $\sum_{i=1}^n v_i \otimes \lambda_i$ of tensor monomials (with varying n). We need not make any assumptions of linear independence or bases! The image

$$T\Big(\sum_{i=1}^n v_i \otimes \lambda_i\Big) \;=\; \sum_{i=1}^n T(v_i \otimes \lambda_i)$$

has range in the span of the v_i. Again, we need not be concerned about linear independence, etc.! Since the sum is finite, that span is finite-dimensional. We need not claim that its dimension is n!

Conversely, given a finite-rank operator S on V, let $v_1, \ldots, v_n$ be a k-basis for the finite-dimensional image of S. Let $\lambda_1, \ldots, \lambda_n \in V^*$ be defined by $\lambda_i(v_j) = 0$ for $i \neq j$, and $\lambda_i(v_i) = 1$. Since $\{v_i\}$ is a basis for the image of S, for every $v \in V$ we have $Sv = \sum_i c_i v_i$ for some scalars c_i, and

$$\lambda_i(Sv) \;=\; \lambda_i\big(\sum_j c_j v_j\big) \;=\; \sum_j \lambda_i(c_j v_j) \;=\; \sum_j c_j \lambda_i(c_j v_j) \;=\; c_i$$

The compositions $\mu_i = \lambda_i \circ S$ are still in V^*, and

$$Sv \;=\; \sum_i \lambda_i(Sv) \cdot v_i \;=\; \sum_i \mu_i(v) \cdot v_i \;=\; T\big(\sum_i \mu_i \cdot v_i \otimes \mu_i\big)(v)$$

This expresses finite-rank S as something in the image of $V \otimes V^*$. ///

[0.2] Remark: An obvious extension of the previous argument shows that $W \otimes_k V^*$ gives all finite-rank k-linear operators from one k-vectorspace V to another one, W. The argument also shows that all rank-one linear maps come from *monomial tensors* $v \otimes \lambda$.

[08.3] *(Coordinate-independent expression for trace)* In the situation of the previous example, let $\Phi_k(V)$ be the finite-rank endomorphisms of V, and $T : \Phi_k(V) \to V \otimes V^*$ the inverse of the map given there. Let $\beta : V \times V^*$ be the bilinear k-valued map $v \times \lambda \to \lambda(v)$, inducing a unique linear k-valued map $B : V \otimes V^*$ specified completely by $B(v \otimes \lambda) = \lambda(v)$. Show that $B \otimes T : \Phi_k(B) \to k$ is the *trace* map on finite-rank endomorphisms.

Discussion: Part of the issue is a suitable *characterization* of trace. The in-coordinates description of trace of a matrix M with entries M_{ij} is $\sum_i M_{ii}$ is not a very good explanation of what trace *is*. For example, it does not explain why trace is *independent of coordinates*. If we can describe/characterize trace of endomorphisms unambiguously *without* coordinates, then we know *a priori* that it is *independent* of coordinates. From this it would *follow* that trace of a square matrix M is equal to trace of AMA^{-1}, and so on.

One possible approach would be to say that trace of a finite-rank endomorphism is the sum of its *eigenvalues*. This certainly works strictly in finite dimensions over an algebraically closed field. And with a touch of cleverness it succeeds for finite-rank operators over an algebraically closed field. But it is simpler than that, insofar as algebraically closed scalars is not a necessity.

For simplicity of notation, let $v \otimes \lambda$ denote the corresponding (rank-one) endomorphism $w \to \lambda(w) \cdot v$. Requiring *linearity* of trace entails that it is uniquely determined by its values on rank-one endomorphisms,

since the formula of the previous exercise incidentally shows that every finite-rank endomorphism is a sum (probably in several different ways) of rank-one endomorphisms.

But, still, why is $\mathrm{tr}(v \otimes \lambda) = \lambda(v)$ a good definition? Taking up the sum-of-eigenvalues idea, we observe that v is the only eigenvector of $v \otimes \lambda$, and has eigenvalue $\lambda(v)$. Because the endomorphism $v \otimes \lambda$ is rank-one, an assumption that the scalars are algebraically closed is not necessary.

Similarly, all other *properties/characterizations* of *trace*, at least for finite-rank operators, follow from $\mathrm{tr}(v \otimes \lambda) = \lambda(v)$. ///

[08.4] Show that there is no continuous extension of *trace* from finite-rank operators on an infinite-dimensional Hilbert space to all continuous operators, and not even to all Hilbert-Schmidt operators.

Discussion: It suffices to show that there is no continuous extension to Hilbert-Schmidt operators. Indeed, the finite-rank operators on ℓ^2 given by

$$T_n(c_1, c_2, c_3, \ldots) \;=\; (\frac{c_1}{1}, \frac{c_2}{2}, \frac{c_1}{3}, \ldots, \frac{c_n}{n}, 0, 0, \ldots)$$

have operator-norm limit

$$T(c_1, c_2, c_3, \ldots) \;=\; (\frac{c_1}{1}, \frac{c_2}{2}, \frac{c_1}{3}, \ldots, \frac{c_n}{n}, \ldots)$$

which is Hilbert-Schmidt, since $\sum 1/n^2 < +\infty$. But $\sum 1/n = +\infty$, so it is not trace class. ///

[08.5] Determine in which Sobolev space(s) $H^s(\mathbb{T}^2)$ the Schwartz kernel for the inclusion $T : \mathcal{D}(\mathbb{T}) \to \mathcal{D}(\mathbb{T})^*$ lies.

Discussion: We treat $\mathcal{D}(\mathbb{T}^n) \to \mathcal{D}(\mathbb{T}^n)^*$ to see the dependence on dimension. Let T be the identity map $\mathcal{D}(\mathbb{T}^n) \to \mathcal{D}(\mathbb{T}^n)$ viewed as a map $\mathcal{D}(\mathbb{T}^n) \to \mathcal{D}(\mathbb{T}^n)^*$ via the natural imbedding $\mathcal{D} \subset \mathcal{D}^*$. Write ψ_ξ for the function $\psi_\xi(x) = e^{2\pi i \xi \cdot x}$ for $\xi \in \mathbb{Z}^n$ and $x \in (\mathbb{R}/\mathbb{Z})^n = \mathbb{T}^n$. Anticipating that there *is* a Schwartz kernel K at worst in $H^{-\infty}(\mathbb{T}^{2n}) = C^\infty(\mathbb{T}^n)$ (the latter equality by Sobolev imbedding), we can write a Fourier expansion $K = \sum_{\xi,\eta \in \mathbb{Z}^n} c_{\xi,\eta}\, \psi_\xi \otimes \psi_\eta$ with coefficients $c_{\xi,\eta}$ to be determined. [1] There is no reason to think that the Fourier series for K converges *pointwise*, and this doesn't matter. The series *does* converge in $H^{-\infty}(\mathbb{T}^{2n})$. The Schwartz kernel for $T : \mathcal{D} \to \mathcal{D}^*$ is *characterized* by

$$K(\varphi \otimes Tf) \;=\; (Tf)(\varphi) \qquad\qquad (\text{for all } \varphi \in \mathcal{D})$$

Applying this to $\varphi = \psi_\alpha$ and $f = \psi_\beta$,

$$c_{\alpha,\beta} \;=\; K(\psi_\alpha \otimes \psi_\beta) \;=\; (T\psi_\beta)(\psi_\alpha) \;=\; \int_{\mathbb{T}^n} \psi_\beta \cdot \psi_\alpha \;=\; \begin{cases} 0 & (\text{for } \beta \neq -\alpha \in \mathbb{Z}^n) \\ 1 & (\text{for } \beta = \alpha \in \mathbb{Z}^n) \end{cases}$$

The latter *necessary* condition already completely determines K: apparently $K = \sum_\alpha \psi_\alpha \otimes \psi_{-\alpha}$. However, we should give a reason why this expression really does give the identity map on $\mathcal{D}(\mathbb{T}^n)$. Certainly

$$\Big| \sum_{\alpha \in \mathbb{Z}^n} \psi_\alpha \otimes \psi_{-\alpha} \Big|^2_{H^s} \;=\; \sum_{\alpha \in \mathbb{Z}^n} |1|^2 \cdot (1 + |\alpha|^2)^s$$

is finite if and only if $s < -\frac{n}{2}$. Thus, for every $\varepsilon > 0$, $K \in H^{-\frac{n}{2}-\varepsilon}(\mathbb{T}^{2n}) \subset H^{-\infty}(\mathbb{T}^{2n}) = H^\infty(\mathbb{T}^{2n})^*$. That is, that Fourier expansion converges in a Sobolev space and does give a distribution on $\mathbb{T}^{2n}$.

[1] The tensor notation here is just a way to refer to the function $x, y \to \psi_\xi(x) \cdot \psi_\eta(y)$ without using arguments.

Since finite linear combinations of ψ_α are *dense* in $\mathcal{D}(\mathbb{T}^n)$, and since K is continuous on $H^\infty(\mathbb{T}^n)\otimes H^\infty(\mathbb{T}^n) \subset H^\infty(\mathbb{T}^{2n})$, the earlier computation of $K(\psi_\alpha \otimes \psi_\beta)$ extends by continuity to certify that $K(f\otimes g) = \int f\cdot g$ for $f, g \in \mathcal{D}(\mathbb{T}^n)$. ///

[08.6] Determine in which Sobolev space(s) $H^s(\mathbb{T}^2)$ the Schwartz kernel for the differentiation map $\frac{d}{dx} : \mathcal{D}(\mathbb{T}) \to \mathcal{D}(\mathbb{T})^*$ lies.

Discussion: The Schwartz kernel theorem assures us that $d/dx : \mathcal{E}(\mathbb{T}) \longrightarrow \mathcal{E}(T)^*$ has a kernel $K \in \mathcal{E}(\mathbb{T}\times\mathbb{T})^*$, characterized by

$$K(\varphi\otimes f) \;=\; (\frac{d}{dx}f)(\varphi) \qquad\qquad (\text{for all } \varphi, f \in \mathcal{E}(\mathbb{T}))$$

Applying this to $\varphi = \psi_m$ and $f = \psi_n$,

$$c_{m,n} \;=\; K(\psi_m\otimes\psi_n) \;=\; (\frac{d}{dx}\psi_n)(\psi_m) \;=\; \int_{\mathbb{T}} 2\pi i n\cdot\psi_n\cdot\psi_m \;=\; \begin{cases} 0 & (\text{for } m \neq -n \in \mathbb{Z}^n) \\ 2\pi i n & (\text{for } m = -n \in \mathbb{Z}^n) \end{cases}$$

Thus,

$$K \;=\; \sum_n 2\pi i n\cdot\psi_{-n}\otimes\psi_n \;\sim\; \sum_n 2\pi i n\cdot e^{2\pi i n(y-x)} \qquad\qquad (\text{convergent in } H^{-\infty}(\mathbb{T}^2))$$

The sth Sobolev norm

$$|K|^2_{H^s(\mathbb{T}^2)} \;=\; \sum_n |2\pi i n|^2\cdot(1+n^2+n^2)^s$$

is finite for $s < -3/2$, so $K \in H^{-3/2-\varepsilon}(\mathbb{T}^2)$ for every $\varepsilon > 0$. ///

[08.7] Determine the Schwartz kernel for the Fourier-Plancherel transform $F : L^2(\mathbb{R}) \to L^2(\mathbb{R})$.

Discussion: It is reasonable to imagine that it is $K(x,y) = e^{-2\pi i\xi x}$, since this is what we integrate against to take the Fourier transform of L^1 functions. However, we know that that integral does not literally converge for all L^2 functions, requiring the extension via Plancherel's theorem.

Thus, it is better to fall back to changing the question to Fourier transform as continuous map $\mathcal{D}(\mathbb{R}) \to \mathcal{D}(\mathbb{R})^*$ and/or $\mathscr{S}(\mathbb{R}) \to \mathscr{S}(\mathbb{R})^*$. With that modification, the integral giving the Fourier transform does literally converge. Also, $K(x,y) = e^{-2\pi i\xi x}$ does give a tempered distribution on $\mathbb{R}^2$. In fact, it is in the Sobolev space $H^{-1-\varepsilon}(\mathbb{R}^2)$ for all $\varepsilon > 0$:

$$\int_{\mathbb{R}^2} |e^{-2\pi i\xi x}|^2\cdot(1+x^2+\xi^2)^s\;dx\,d\xi \;=\; \int_{\mathbb{R}^2}(1+x^2+\xi^2)^s\;dx\,d\xi$$

which is absolutely convergent for all $s < -1$. ///

[08.8] Show that an operator $T : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ given by a Schwartz kernel $K(x,y)$ in $H^{\frac{1}{2}+\varepsilon}(\mathbb{T}\times\mathbb{T})$ is trace-class. Show that $x \to K(x,x)$ is in $L^2(\mathbb{T})$. Show that T has trace $\operatorname{tr} T = \int_{\mathbb{T}} K(x,x)\,dx$. (*Hint:* Trace theorem... with different sense of *trace*.)

Discussion: The *trace theorem* implies that the restriction to the diagonal copy of $\mathbb{T}$ is in $H^{\varepsilon-\varepsilon'}(\mathbb{T})$ for every $\varepsilon' > 0$, so certainly is in $L^2(\mathbb{T}) \subset L^1(\mathbb{T})$.

To prove that K gives a trace-class operator T, we express T as a composition $T = R\circ S$ of two Hilbert-Schmidt operators. If this is possible at all, then there will be many ways to do it, and we must make simplifying choices. For two kernels $R, S \in \mathcal{D}(\mathbb{T})^* = H^{-\infty}(\mathbb{T}^2)$, the kernel of the composition $R\circ S$ of the operators is readily computed in terms of the kernels of R and S:

$$\text{kernel}\, R\circ S \;=\; \sum_{k,n}\Big(\sum_\ell \widehat{R}(k,\ell)\,\widehat{S}(-\ell,n)\Big)\cdot\psi_k\otimes\psi_n$$

We want R, S Hilbert-Schmidt, that is, we want

$$\sum_{m,n} |\widehat{R}(m,n)|^2 \;<\; +\infty \qquad\qquad \sum_{m,n} |\widehat{S}(m,n)|^2 \;<\; +\infty$$

We make a slightly clever choice:

$$\widehat{S}(m,n) \;=\; \begin{cases} (1+n^2)^{-(\frac{1}{2}+\varepsilon)/2} & (\text{for } m=-n) \\ \\ 0 & (\text{for } m \not\sim n) \end{cases}$$

and $\widehat{R}(m,n) = \widehat{K} \cdot (1+n^2)^{(\frac{1}{2}+\varepsilon)/2}$. By Plancherel, $S \in L^2(\mathbb{T}^2)$. And

$$\sum_{m,n} |\widehat{R}_{m,n}|^2 \;=\; \sum_{m,n} |\widehat{K}_{m,n}|^2 \cdot (1+n^2)^{\frac{1}{2}+\varepsilon} \;\le\; \sum_{m,n} |\widehat{K}_{m,n}|^2 \cdot (1+m^2+n^2)^{\frac{1}{2}+\varepsilon} \;<\; +\infty$$

By Plancherel, $R \in L^2(\mathbb{T}^2)$. This expresses the operator given by kernel K as a composition of two Hilbert-Schmidt operators, so it is trace-class, indeed.

And, then, ignoring for simplicity issues of complex conjugation, its trace is the Hilbert-Schmidt inner product of the two kernels: by Plancherel,

$$\operatorname{tr} K \;=\; \langle R, S\rangle \;=\; \sum_{m,n} \widehat{R}(m,n)\cdot\widehat{S}(m,n) \;=\; \sum_{m=n} \widehat{K}(m,n)\cdot(1+n^2)^{(\frac{1}{2}+\varepsilon)/2}\cdot(1+n^2)^{-(\frac{1}{2}+\varepsilon)/2} \;=\; \sum_n \widehat{K}(n,n)$$

as we imagined it should be. ///

[08.9] Let

$$K(x,y) \;=\; \begin{cases} x\cdot(\frac{y}{2\pi}-1) & (\text{in } 0<x<y) \\ \\ y\cdot(\frac{x}{2\pi}-1) & (\text{in } 2\pi>x>y) \end{cases}$$

be the kernel for a continuous linear map $T : L^2[0,2\pi] \to L^2[0,2\pi]$. We have seen that T is *compact* and self-adjoint, with eigenvectors $\sin\frac{nx}{2}$, for $n = 1,2,3,\ldots$. Show that T is trace-class. Take its trace to give yet another proof that

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots \;=\; \frac{\pi^2}{6}$$

Discussion: The operator is compact, self-adjoint, so $L^2[0,2\pi]$ has an orthogonal basis of eigenvectors. The eigenvalues for the differential operator Δ are $-(n/2)^2$, so the eigenvalues for the integral operator are $-4/n^2$, with multiplicity 1. At least if the operator is genuinely trace-class, its trace is $-4\sum_n \frac{1}{n^2}$.

On the other hand, at least heuristically, the trace should also be the integral of the kernel along the diagonal:

$$\operatorname{tr}(T) \;=\; \int_0^{2\pi} K(x,x)\,dx \;=\; \int_0^{2\pi} x(\frac{x}{2\pi}-1)\,dx \;=\; \Big[\frac{x^3}{6\pi}-\frac{x^2}{2}\Big]_0^{2\pi} \;=\; \frac{8\pi^3}{6\pi}-\frac{4\pi^2}{2} \;=\; -\frac{2}{3}\pi^2$$

Equating the two expressions for trace,

$$-4\sum_n \frac{1}{n^2} \;=\; -\frac{2}{3}\pi^2$$

so

$$\sum_n \frac{1}{n^2} \;=\; \frac{\pi^2}{6}$$

as we anticipated. ///
