Examples Discussion 08[draft]

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08.1  For abelian groups $A, B, C$, prove that $\text{Hom}(A \otimes B, C) \approx \text{Hom}(A, \text{Hom}(B, C))$.

Discussion: This isomorphism (together with its nice properties) is the adjunction given by

$$\Phi \rightarrow \varphi \Phi \quad \text{with} \quad \varphi \Phi(a)(b) = \Phi(a \otimes b) \quad \text{and} \quad \Phi \varphi \leftarrow \varphi \quad \text{with} \quad \Phi \varphi(a \otimes b) = \varphi(a)(b)$$

The map $\Phi \varphi \leftarrow \varphi$ uses properties of the tensor product. Specifically, $\beta \varphi(a \times b) = \varphi(a)(b)$ makes immediate sense, and by properties of the tensor product $A \otimes B$ of abelian groups, a bilinear map $\beta \varphi : A \times B \rightarrow C$ produces a unique linear map $\Phi \varphi : A \otimes B \rightarrow C$ through which $\beta \varphi$ factors.

The latter remarks show that $\Phi \varphi$ really is in $\text{Hom}(A \otimes B, C)$. Similarly, $\varphi \Phi$ really is in $\text{Hom}(A, \text{Hom}(B, C))$, that is, it is a linear map from $A$ to $\text{Hom}(B, C)$: indeed

$$\varphi \Phi(a + a')(b) = \Phi((a + a') \otimes b) = \Phi(a \otimes b) + \Phi(a' \otimes b) = \varphi \Phi(a)(b) + \varphi \Phi(a')(b)$$

The maps are mutual inverses, at the level of sets: in one direction, for $\Psi \in \text{Hom}(A \otimes B, C)$,

$$\Phi \varphi \Phi(a \otimes b) = \varphi \Phi(a)(b) = \Psi(a \otimes b)$$

Similarly, in the other direction, for $\psi \in \text{Hom}(A, \text{Hom}(B, C))$,

$$\varphi \psi \psi(a)(b) = \Phi \psi(a \otimes b) = \psi(a)(b)$$

Additivity of $\Phi \varphi \leftarrow \varphi$: for $\psi \in \text{Hom}(A, \text{Hom}(B, C))$:

$$\Phi \varphi + \psi(a \otimes b) = (\varphi + \psi)(a)(b) = \varphi(a)(b) + \psi(a)(b) = (\Phi \varphi + \Phi \psi)(a \otimes b)$$

Additivity in the opposite direction is similar.  

0.1  Remark: The naturality aspect of the adjunction, roughly, that the adjunction behaves well as $A, B, C$ vary, does also require proof, but we are not using that aspect at the moment.

08.2  For a field $k$ and a $k$-vector space $V$ (without topology), show that the map $V \otimes V^* \rightarrow \text{End}_k V$ induced from $(v \otimes \lambda)(w) = \lambda(w) \cdot v$, for $v, w \in V$ and $\lambda \in V^*$, is a bijection to finite-rank endomorphisms of $V$ (meaning that their images are finite-dimensional).

Discussion: Part of the point is that this is true over arbitrary fields $k$, and does not depend on having an inner product. Let $Tv \otimes \lambda(w) = \lambda(w) \cdot v$ for $v \in V$ and $\lambda \in V^*$.

First, to be sure that the map $T$ exists and is unambiguous, we exactly must show that the map $V \times V^* \rightarrow \text{End}_k V$ is bilinear, from which the existence and uniqueness of $T$ will follow by the properties of tensor products. We carry out these routine verifications, to show how it’s done. Additivity in the first argument:

$$(v + v') \times \lambda)(w) = \lambda(w) \cdot (v + v') = \lambda(w) \cdot v + \lambda(w) \cdot v'$$

and in the second argument:

$$(v \times (\lambda + \lambda'))(w) = (\lambda + \lambda')(w) \cdot v = \lambda(w) \cdot v + \lambda'(w) \cdot v$$
Similarly for scalars $c$:

$$(cv \times \lambda)(w) = \lambda(w) \cdot cv = c \cdot (\lambda(w) \cdot v) = c \cdot ((v \times \lambda)(w))$$

and

$$(v \times c\lambda)(w) = (c\lambda)(w) \cdot v = c \cdot (\lambda(w) \cdot v) = c \cdot ((v \times \lambda)(w))$$

So $T$ exists, and is unambiguous.

Now, more interestingly, we show that $T(V \otimes V^*)$ consists of finite-rank endomorphisms of $V$. Indeed, every element of $V \otimes V^*$ is a finite sum $\sum_{i=1}^n v_i \otimes \lambda_i$ of tensor monomials (with varying $n$). We need not make any assumptions of linear independence or bases! The image

$$T\left(\sum_{i=1}^n v_i \otimes \lambda_i\right) = \sum_{i=1}^n T(v_i \otimes \lambda_i)$$

has range in the span of the $v_i$. Again, we need not be concerned about linear independence, etc! Since the sum is finite, that span is finite-dimensional. We need not claim that its dimension is $n$!

Conversely, given a finite-rank operator $S$ on $V$, let $v_1, \ldots, v_n$ be a $k$-basis for the finite-dimensional image of $S$. Let $\lambda_1, \ldots, \lambda_n \in V^*$ be defined by $\lambda_i(v_j) = 0$ for $i \neq j$, and $\lambda_i(v_i) = 1$. Since $\{v_i\}$ is a basis for the image of $S$, for every $v \in V$ we have $Sv = \sum_i c_i v_i$ for some scalars $c_i$, and

$$\lambda_i(Sv) = \lambda_i(\sum_j c_j v_j) = \sum_j \lambda_i(c_j v_j) = \sum_j c_j \lambda_i(v_j) = c_i$$

The compositions $\mu_i = \lambda_i \circ S$ are still in $V^*$, and

$$Sv = \sum_i \lambda_i(Sv) \cdot v_i = \sum_i \mu_i(v) \cdot v_i = T(\sum_i \mu_i \cdot v_i \otimes \mu_i)(v)$$

This expresses finite-rank $S$ as something in the image of $V \otimes V^*$.

[0.2] **Remark:** An obvious extension of the previous argument shows that $W \otimes_k V^*$ gives all finite-rank $k$-linear operators from one $k$-vectorspace $V$ to another one, $W$. The argument also shows that all rank-one linear maps come from monomial tensors $v \otimes \lambda$.

[08.3] **(Coordinate-independent expression for trace)** In the situation of the previous example, let $\Phi_k(V)$ be the finite-rank endomorphisms of $V$, and $T: \Phi_k(V) \to V \otimes V^*$ the inverse of the map given there. Let $\beta: V \times V^*$ be the bilinear $k$-valued map $v \times \lambda \to \lambda(v)$, inducing a unique linear $k$-valued map $B: V \otimes V^*$ specified completely by $B(v \otimes \lambda) = \lambda(v)$. Show that $B \otimes T: \Phi_k(B) \to k$ is the trace map on finite-rank endomorphisms.

**Discussion:** Part of the issue is a suitable characterization of trace. The in-coordinates description of trace of a matrix $M$ with entries $M_{ij}$ is $\sum_i M_{ii}$ is not a very good explanation of what trace is. For example, it does not explain why trace is independent of coordinates. If we can describe/characterize trace of endomorphisms unambiguously without coordinates, then we know a priori that it is independent of coordinates. From this it would follow that trace of a square matrix $M$ is equal to trace of $AMA^{-1}$, and so on.

One possible approach would be to say that trace of a finite-rank endomorphism is the sum of its eigenvalues. This certainly works strictly in finite dimensions over an algebraically closed field. And with a touch of cleverness it succeeds for finite-rank operators over an algebraically closed field. But it is simpler than that, insofar as algebraically closed scalars is not a necessity.

For simplicity of notation, let $v \otimes \lambda$ denote the corresponding (rank-one) endomorphism $w \to \lambda(w) \cdot v$. Requiring linearity of trace entails that it is uniquely determined by its values on rank-one endomorphisms,
since the formula of the previous exercise incidentally shows that every finite-rank endomorphism is a sum (probably in several different ways) of rank-one endomorphisms.

But, still, why is \( \text{tr}(v \otimes \lambda) = \lambda(v) \) a good definition? Taking up the sum-of-eigenvalues idea, we observe that \( v \) is the only eigenvector of \( v \otimes \lambda \), and has eigenvalue \( \lambda(v) \). Because the endomorphism \( v \otimes \lambda \) is rank-one, an assumption that the scalars are algebraically closed is not necessary.

Similarly, all other properties/characterizations of trace, at least for finite-rank operators, follow from \( \text{tr}(v \otimes \lambda) = \lambda(v) \).

### [08.4] Show that there is no continuous extension of trace from finite-rank operators on an infinite-dimensional Hilbert space to all continuous operators, and not even to all Hilbert-Schmidt operators.

**Discussion:** It suffices to show that there is no continuous extension to Hilbert-Schmidt operators. Indeed, the finite-rank operators on \( l^2 \) given by

\[
T_n(c_1, c_2, c_3, \ldots) = \left( \frac{c_1}{1}, \frac{c_2}{2}, \frac{c_3}{3}, \ldots, \frac{c_n}{n}, 0, 0, \ldots \right)
\]

have operator-norm limit

\[
T(c_1, c_2, c_3, \ldots) = \left( \frac{c_1}{1}, \frac{c_2}{2}, \frac{c_3}{3}, \ldots, \frac{c_n}{n}, \ldots \right)
\]

which is Hilbert-Schmidt, since \( \sum 1/n^2 < +\infty \). But \( \sum 1/n = +\infty \), so it is not trace class.

### [08.5] Determine in which Sobolev space(s) \( H^s(\mathbb{T}^2) \) the Schwartz kernel for the inclusion \( T : D(\mathbb{T}) \to D(\mathbb{T}^*) \) lies.

**Discussion:** We treat \( D(\mathbb{T}^n) \to D(\mathbb{T}^n)^* \) to see the dependence on dimension. Let \( T \) be the identity map \( D(\mathbb{T}^n) \to D(\mathbb{T}^n)^* \) viewed as a map \( D(\mathbb{T}^n) \to D(\mathbb{T}^n)^* \) via the natural imbedding \( D \subset D^* \). Write \( \psi_\xi \) for the function \( \psi_\xi(x) = e^{2\pi i \xi \cdot x} \) for \( \xi \in \mathbb{Z}^n \) and \( x \in (\mathbb{R}/\mathbb{Z})^n = \mathbb{T}^n \). Anticipating that there is a Schwartz kernel \( K \) at worst in \( H^{-\infty}(\mathbb{T}^2n) = C^\infty(\mathbb{T}^n) \) (the latter equality by Sobolev imbedding), we can write a Fourier expansion

\[
K = \sum_{\xi, \eta \in \mathbb{Z}^n} c_{\xi, \eta} \psi_\xi \otimes \psi_\eta
\]

with coefficients \( c_{\xi, \eta} \) to be determined. \(^1\) There is no reason to think that the Fourier series for \( K \) converges pointwise, and this doesn’t matter. The series does converge in \( H^{-\infty}(\mathbb{T}^2n) \).

The Schwartz kernel for \( T : D \to D^* \) is characterized by

\[
K(\varphi \otimes Tf) = (Tf)(\varphi) \quad \text{for all } \varphi \in D
\]

Applying this to \( \varphi = \psi_\alpha \) and \( f = \psi_\beta \),

\[
c_{\alpha, \beta} = K(\psi_\alpha \otimes \psi_\beta) = (T\psi_\beta)(\psi_\alpha) = \int_{\mathbb{T}^n} \psi_\beta \cdot \psi_\alpha = \begin{cases} 0 & \text{for } \beta \neq -\alpha \in \mathbb{Z}^n \\ 1 & \text{for } \beta = \alpha \in \mathbb{Z}^n \end{cases}
\]

The latter necessary condition already completely determines \( K \): apparently \( K = \sum_\alpha \psi_\alpha \otimes \psi_{-\alpha} \). However, we should give a reason why this expression really does give the identity map on \( D(\mathbb{T}^n) \). Certainly

\[
\left\| \sum_{\alpha \in \mathbb{Z}^n} \psi_\alpha \otimes \psi_{-\alpha} \right\|^2_{H^s} = \sum_{\alpha \in \mathbb{Z}^n} |1|^2 \cdot (1 + |\alpha|^2)^s
\]

is finite if and only if \( s < -\frac{n}{2} \). Thus, for every \( \varepsilon > 0 \), \( K \in H^{-\frac{n}{2}+\varepsilon}(\mathbb{T}^2n) \subset H^{-\infty}(\mathbb{T}^2n) = H^{\infty}(\mathbb{T}^2n)^* \). That is, that Fourier expansion converges in a Sobolev space and does give a distribution on \( \mathbb{T}^2n \).

\(^1\) The tensor notation here is just a way to refer to the function \( x, y \to \psi_\xi(x) \cdot \psi_\eta(y) \) without using arguments.
Since finite linear combinations of $\psi_\alpha$ are dense in $D(T^n)$, and since $K$ is continuous on $H^\infty(T^n) \otimes H^\infty(T^n) \subset H^\infty(T^{2n})$, the earlier computation of $K(\psi_\alpha \otimes \psi_\beta)$ extends by continuity to certify that $K(f \otimes g) = \int f \cdot g$ for $f, g \in D(T^n)$.

\[08.6\] Determine in which Sobolev space(s) $H^s(T^2)$ the Schwartz kernel for the differentiation map $\frac{d}{dx} : D(T) \to D(T)^*$ lies.

**Discussion:** The Schwartz kernel theorem assures us that $\frac{d}{dx}$ has a trace-class operator $T$ with different sense of trace theorem... with different sense of trace. Thus, $T$ gives a trace-class operator $\mathcal{T}$ characterized by

$$\mathcal{T}(\varphi) = \int (d/dx) \varphi.$$ 

Applying this to $\varphi = \psi_m$ and $f = \psi_n$,

$$c_{m,n} = K(\psi_m \otimes \psi_n) = \int 2\pi i n \cdot \psi_n \cdot \psi_m = \begin{cases} 0 & \text{(for } m \neq -n \in \mathbb{Z}^n) \\ 2\pi i n & \text{(for } m = -n \in \mathbb{Z}^n) \end{cases}$$

Thus,

$$K = \sum_n 2\pi i n \cdot \psi_n \otimes \psi_n \sim \sum_n 2\pi i n \cdot e^{2\pi i n(y-x)}$$

is finite for $s < -3/2$, so $K \in H^{-3/2-\varepsilon}(T^2)$ for every $\varepsilon > 0$.

\[08.7\] Determine the Schwartz kernel for the Fourier-Plancherel transform $F : L^2(\mathbb{R}) \to L^2(\mathbb{R})$.

**Discussion:** It is reasonable to imagine that it is $K(x, y) = e^{-2\pi i x \cdot y}$, since this is what we integrate against to take the Fourier transform of $L^1$ functions. However, we know that that integral does not literally converge for all $L^2$ functions, requiring the extension via Plancherel’s theorem.

Thus, it is better to fall back to changing the question to Fourier transform as continuous map $D(\mathbb{R}) \to D(\mathbb{R})^*$ and/or $\mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})^*$. With that modification, the integral giving the Fourier transform does literally converge. Also, $K(x, y) = e^{-2\pi i \xi \cdot y}$ does give a tempered distribution on $\mathbb{R}^2$. In fact, it is in the Sobolev space $H^{-1-\varepsilon}(\mathbb{R}^2)$ for all $\varepsilon > 0$:

$$\int_{\mathbb{R}^2} \left| e^{-2\pi i \xi \cdot y} \right|^2 \cdot (1 + x^2 + \xi^2)^s \, dx \, d\xi = \int_{\mathbb{R}^2} (1 + x^2 + \xi^2)^s \, dx \, d\xi$$

which is absolutely convergent for all $s < -1$.

\[08.8\] Show that an operator $T : L^2(T) \to L^2(T)$ given by a Schwartz kernel $K(x, y)$ in $H^{1+\varepsilon}(T \times T)$ is trace-class. Show that $x \to K(x, x)$ is in $L^2(T)$. Show that $T$ has trace $\text{tr} T = \int_T K(x, x) \, dx$. (Hint: Trace theorem... with different sense of trace.)

**Discussion:** The trace theorem implies that the restriction to the diagonal copy of $T$ is in $H^{s-\varepsilon'}(T)$ for every $\varepsilon' > 0$, so certainly is in $L^2(T) \subset L^1(T)$. To prove that $K$ gives a trace-class operator $T$, we express $T$ as a composition $T = R \circ S$ of two Hilbert-Schmidt operators. If this is possible at all, then there will be many ways to do it, and we must make simplifying choices. For two kernels $R, S \in D(T)^* = H^{-\infty}(T^2)$, the kernel of the composition $R \circ S$ of the operators is readily computed in terms of the kernels of $R$ and $S$:

$$\text{kernel } R \circ S = \sum_{k,n} \sum_{\ell} \hat{R}(k, \ell) \hat{S}(-\ell, n) \cdot \psi_k \otimes \psi_n$$
We want $R, S$ Hilbert-Schmidt, that is, we want
\[ \sum_{m,n} |\hat{R}(m,n)|^2 < +\infty \quad \sum_{m,n} |\hat{S}(m,n)|^2 < +\infty \]

We make a slightly clever choice:
\[ \hat{S}(m,n) = \begin{cases} (1 + n^2)^{-\frac{1}{2} + \varepsilon/2} & \text{(for } m = -n) \\ 0 & \text{(for } m \neq -n) \end{cases} \]
and $\hat{R}(m,n) = \hat{K} \cdot (1 + n^2)^{\frac{1}{2} + \varepsilon/2}$. By Plancherel, $S \in L^2(\mathbb{T}^2)$. And
\[ \sum_{m,n} |\hat{R}_{m,n}|^2 = \sum_{m,n} |\hat{K}_{m,n}|^2 \cdot (1 + n^2)^{\frac{1}{2} + \varepsilon} \leq \sum_{m,n} |\hat{K}_{m,n}|^2 \cdot (1 + m^2 + n^2)^{\frac{1}{2} + \varepsilon} < +\infty \]
By Plancherel, $R \in L^2(\mathbb{T}^2)$. This expresses the operator given by kernel $K$ as a composition of two Hilbert-Schmidt operators, so it is trace-class, indeed.

And, then, ignoring for simplicity issues of complex conjugation, its trace is the Hilbert-Schmidt inner product of the two kernels: by Plancherel,
\[ \text{tr } K = \langle R, S \rangle = \sum_{m,n} \hat{R}(m,n) \cdot \hat{S}(m,n) = \sum_{m=n} \hat{K}(m,n) \cdot (1 + n^2)^{(\frac{1}{2} + \varepsilon)/2} \cdot (1 + n^2)^{-\frac{1}{2} - \varepsilon} = \sum_n \hat{K}(n,n) \]
as we imagined it should be. \///

**[08.9]** Let
\[ K(x,y) = \begin{cases} x \cdot \left( \frac{x}{2\pi} - 1 \right) & \text{(in } 0 < x < y) \\ y \cdot \left( \frac{y}{2\pi} - 1 \right) & \text{(in } 2\pi > x > y) \end{cases} \]
be the kernel for a continuous linear map $T : L^2[0,2\pi] \to L^2[0,2\pi]$. We have seen that $T$ is compact and self-adjoint, with eigenvectors $\sin \frac{nx}{2\pi}$, for $n = 1, 2, 3, \ldots$. Show that $T$ is trace-class. Take its trace to give yet another proof that
\[ \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots = \frac{\pi^2}{6} \]

**Discussion:** The operator is compact, self-adjoint, so $L^2[0,2\pi]$ has an orthogonal basis of eigenvectors. The eigenvalues for the differential operator $\Delta$ are $-\left(\frac{n}{2}\right)^2$, so the eigenvalues for the integral operator are $-4/n^2$, with multiplicity 1. At least if the operator is genuinely trace-class, its trace is $-4 \sum_n \frac{1}{n^2}$.

On the other hand, at least heuristically, the trace should also be the integral of the kernel along the diagonal:
\[ \text{tr}(T) = \int_0^{2\pi} K(x,x) \, dx = \int_0^{2\pi} x \left( \frac{x}{2\pi} - 1 \right) \, dx = \left[ \frac{x^3}{6\pi} - \frac{x^2}{2} \right]_0^{2\pi} = \frac{8\pi^3}{6\pi} - \frac{4\pi^2}{2} = -\frac{2}{3} \pi^2 \]

Equating the two expressions for trace,
\[ -4 \sum_n \frac{1}{n^2} = -\frac{2}{3} \pi^2 \]
so
\[ \sum_n \frac{1}{n^2} = \frac{\pi^2}{6} \]
as we anticipated. \///