Examples 03

[This document is http://www.math.umn.edu/~garrett/m/real/notes_2019-20/real-ex-03.pdf]

For feedback on these examples, please get your write-ups to me by Friday, 08 Nov 2019.

[03.1] For a vector subspace \( W \) of a Hilbert space \( V \), show that \((W^\perp)^\perp\) is the topological closure of \( W \).

[03.2] Find two dense vector subspaces \( X, Y \) of \( \ell^2 \) such that \( X \cap Y = \{0\} \). (And, if you need further entertainment, can you find countably-many dense vector subspaces \( X_n \) such that \( X_m \cap X_n = \{0\} \) for \( m \neq n \)?)

[03.3] For measurable \( E \subseteq [0,1] \), show that \( \lim_{n \to \infty} \int_E e^{-2\pi i nx} \, dx = 0 \) as \( n \to \infty \) ranging over integers.

[03.4] Let \( f_n(x) = \sin \pi nx \) on \([0,1]\), extended by \( \mathbb{Z} \)-periodicity, for \( n = 1, 2, 3, \ldots \). Given \( g \in L^1[0,1] \), show that \( \int_0^1 f_n \cdot g \to 0 \).

[03.5] Compute the Fourier coefficients of the sawtooth function \( s(x) = x - \frac{1}{2} \) on \([0,1]\), extended by \( \mathbb{Z} \)-periodicity. Use this to show that \( \sum_{n \geq 1} 1/n^2 = \pi^2/6 \).

[03.6] Let \( E \) be a Lebesgue measurable set in \( \mathbb{R} \) with finite Lebesgue measure. Show that
\[
\lim_{t \to +\infty} \int_E \sin tx \, dx = 0 \quad \text{(over real } t)\]

[03.7] Compute \( \int_{\mathbb{R}} \left(\frac{\sin x}{x}\right)^2 \, dx \). (Hint: do not attempt to do this directly, nor by complex analysis.)

[03.8] (Collecting Fourier transform pairs) Compute the Fourier transforms of
\[
\chi_{[a,b]} \quad e^{-\pi x^2} \quad f(x) = \begin{cases} e^{-x} & \text{(for } x > 0) \\ 0 & \text{(for } x \leq 0) \end{cases}
\]

[03.9] Give an explicit non-zero function \( f \) such that \( \int_{\mathbb{R}} x^n f(x) \, dx = 0 \), for all \( n = 0, 1, 2, \ldots \).

[03.10] Show that \( \chi_{[a,b]} * \chi_{[c,d]} \) is a piecewise-linear function, and express it explicitly.

[03.11] For \( f \in \mathcal{S} \), show that
\[
\lim_{\varepsilon \to 0^+} f(x) * \frac{e^{-\pi x^2/\varepsilon}}{\sqrt{\varepsilon}} = f(x)
\]

[03.12] For \( f \in \mathcal{S} \), show that
\[
\lim_{t \to +\infty} f(x) * \frac{2\sin tx}{tx} = f(x)
\]

[03.13] Evaluate the Borwein integral
\[
\int_{\mathbb{R}} \frac{\sin x}{x} \cdot \frac{\sin x/3}{x/3} \cdot \frac{\sin x/5}{x/5} \, dx
\]