Examples 03

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[This document is http://www.math.umn.edu/~garrett/m/real/notes_2019-20/real-ex-03.pdf]

For feedback on these examples, please get your write-ups to me by Friday, 08 Nov 2019.

[03.1] For a vector subspace $W$ of a Hilbert space $V$, show that $(W^\perp)^\perp$ is the topological closure of $W$.

[03.2] Find two dense vector subspaces $X, Y$ of $\ell^2$ such that $X \cap Y = \{0\}$. (And, if you need further entertainment, can you find countably-many dense vector subspaces $X_n$ such that $X_m \cap X_n = \{0\}$ for $m \neq n$?)

[03.3] For measurable $E \subset [0, 1]$, show that $\lim_n \int_E e^{-2\pi i nx} \, dx = 0$ as $n \to \infty$ ranging over integers.

[03.4] Let $f_n(x) = \sin \pi nx$ on $[0, 1]$, extended by $\mathbb{Z}$-periodicity, for $n = 1, 2, 3, \ldots$. Given $g \in L^1[0, 1]$, show that $\int_0^1 f_n \cdot g \to 0$.

[03.5] Compute the Fourier coefficients of the sawtooth function $s(x) = x - \frac{1}{2}$ on $[0, 1]$, extended by $\mathbb{Z}$-periodicity. Use this to show that $\sum_{n \geq 1} 1/n^2 = \pi^2/6$.

[03.6] Let $E$ be a Lebesgue measurable set in $\mathbb{R}$ with finite Lebesgue measure. Show that

$$\lim_{t \to +\infty} \int_E \sin tx \, dx = 0 \quad \text{(over real } t)$$

[03.7] Compute $\int_{\mathbb{R}} \left( \frac{\sin x}{x} \right)^2 \, dx$. (Hint: do not attempt to do this directly, nor by complex analysis.)

[03.8] (Collecting Fourier transform pairs) Compute the Fourier transforms of

$$\chi_{[a,b]} \quad e^{-\pi x^2} \quad f(x) = \begin{cases} e^{-x} & \text{(for } x > 0) \\ 0 & \text{(for } x \leq 0) \end{cases}$$

[03.9] Give an explicit non-zero function $f$ such that $\int_{\mathbb{R}} x^n f(x) \, dx = 0$, for all $n = 0, 1, 2, \ldots$.

[03.10] Show that $\chi_{[a,b]} \star \chi_{[c,d]}$ is a piecewise-linear function, and express it explicitly.

[03.11] For $f \in \mathcal{S}$, show that

$$\lim_{\varepsilon \to 0^+} f(x) \ast \frac{e^{-\pi x^2/\varepsilon}}{\sqrt{\varepsilon}} = f(x)$$

[03.12] (Corrected!) For $f \in \mathcal{S}$, show that

$$\lim_{t \to +\infty} f(x) \ast \frac{\sin 2\pi tx}{\pi x} = f(x)$$
Evaluate the Borwein integral

\[ \int_{\mathbb{R}} \frac{\sin x}{x} \cdot \frac{\sin x/3}{x/3} \cdot \frac{\sin x/5}{x/5} \, dx \]