For feedback on these examples, please get your write-ups to me by Friday, April 24, 2020.

[08.1] For abelian groups $A, B, C$, prove that $\text{Hom}(A \otimes B, C) \approx \text{Hom}(A, \text{Hom}(B, C))$.

[08.2] For a field $k$ and a $k$-vectorspace $V$ (without topology), show that the map $V \otimes V^* \to \text{End}_k V$ induced from $(v \otimes \lambda)(w) = \lambda(w) \cdot v$, for $v, w \in V$ and $\lambda \in V^*$, is a bijection to finite-rank endomorphisms of $V$ (meaning that their images are finite-dimensional).

[08.3] (Coordinate-independent expression for trace) In the situation of the previous example, let $\Phi_k(V)$ be the finite-rank endomorphisms of $V$, and $T : \Phi_k(V) \to V \otimes V^*$ the inverse of the map given there. Let $\beta : V \times V^* \to \mathbb{R}$ be the bilinear $k$-valued map $v \times \lambda \to \lambda(v)$, inducing a linear $k$-valued map $B : V \otimes V^*$ given by $B(v \otimes \lambda) = \lambda(v)$. Show that $B \otimes T : \Phi_k(B) \to k$ is the trace map on finite-rank endomorphisms.

[08.4] Show that there is no continuous extension of trace from finite-rank operators on an infinite-dimensional Hilbert space to all continuous operators, and not even to all Hilbert-Schmidt operators.

[08.5] Determine in which Sobolev space(s) $H^s(\mathbb{R}^2)$ the Schwartz kernel for the inclusion $T : \mathcal{D}(\mathbb{T}) \to \text{test}(\mathbb{T})^*$ lies.

[08.6] Determine in which Sobolev space(s) $H^s(\mathbb{R}^2)$ the Schwartz kernel for the differentiation map $\frac{d}{dx} : \mathcal{D}(\mathbb{T}) \to \mathcal{D}(\mathbb{T})^*$ lies.

[08.7] Determine the Schwartz kernel for the Fourier-Plancherel transform $F : L^2(\mathbb{R}) \to L^2(\mathbb{R})$.

[08.8] Show that an operator $T : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ given by a Schwartz kernel $K(x, y)$ in $H^{\frac{1}{2}}[0, 2\pi]$ is trace-class. Show that $x \to K(x, x)$ is in $L^2(\mathbb{T})$. Show that $T$ has trace $\text{tr} T = \int_\mathbb{T} K(x, x) \, dx$. (Hint: Trace theorem... with different sense of trace.)

[08.9] Let
\[
K(x, y) = \begin{cases} 
  x \cdot \left( \frac{y}{2\pi} - 1 \right) & \text{(in } 0 < x < y) \\
  y \cdot \left( \frac{x}{2\pi} - 1 \right) & \text{(in } 2\pi > x > y) 
\end{cases}
\]
be the kernel for a continuous linear map $T : L^2[0, 2\pi] \to L^2[0, 2\pi]$. We have seen that $T$ is compact and self-adjoint, with eigenvectors $\sin n x$, for $n = 1, 2, 3, \ldots$. Show that $T$ is trace-class. Take its trace to give yet another proof that
\[
\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6}
\]