02b. **Riesz-Markov-Kakutani theorem, Lebesgue measure**

Paul Garrett  garrett@math.umn.edu  http://www.math.umn.edu/~garrett/

[This document is http://www.math.umn.edu/~garrett/m/real/notes-2016-17/real-notes-02b.pdf]

1. Riesz-Markov-Kakutani theorem and regularity
2. Lebesgue measure

---

### 1. Riesz-Markov-Kakutani theorem and regularity

Let $X$ be a locally compact, Hausdorff topological space. A map $f \rightarrow \lambda(f)$ of continuous, compactly supported functions $C_c^0(X)$ to scalars is positive when $\lambda(f) \geq 0$ for $f \in C_c^0(X)$ taking values in $[0, +\infty)$.

**[1.1] Theorem:** *(Riesz, Markov, Kakutani, independently)* Given a positive functional $\lambda$ on $C_c^0(X)$, there is a $\sigma$-algebra $A$ containing all Borel sets, and a positive measure $\mu$ on $A$, such that

$$\lambda(f) = \int_X f(x) \, d\mu(x) \quad \text{(for all } f \in C_c^0(X))$$

- **Outer regularity** holds unconditionally, namely, that for $E \in A$, $\mu(E) = \inf_{U \supseteq E} \mu(U)$ where $U$ ranges over open sets containing $E$.
- **Inner regularity** is conditional: for open $E$, and for $\mu(E) < \infty$, $\mu(E) = \sup_{K \subseteq E} \mu(K)$ where $K$ ranges over compact sets contained in $E$.
- $\mu$ is complete, in the sense that $E' \subseteq E \in A$ and $\mu(E) = 0$ implies that $E' \in A$.

**Proof:** (Standard... [... iou ...])

With a further mild assumption on the physical space $X$, including familiar spaces such as $\mathbb{R}^n$, in fact we have unconditional regularity.

**[1.2] Theorem:** Suppose further that $X$ is $\sigma$-compact, meaning that it is a countable union of compact subsets. Then, in the situation of the previous theorem, $\mu$ is unconditionally inner regular: $\mu(E) = \sup_{K \subseteq E} \mu(K)$ as $K$ ranges over compacts contained in $E$. Thus, the measure $\mu$ is a positive, regular, Borel measure.

**Proof:** (Standard... [... iou ...])

---

### 2. Lebesgue measure

As a corollary of the Riesz-Markov-Kakutani theorem we have a different description of the Lebesgue measure and integral, as an extension of the Riemann integral, with the very useful side effect of proving inner and outer regularity.

In the Riesz-Markov-Kakutani theorem, take $X = \mathbb{R}^n$, and $\lambda(f)$ to be the usual Riemann integral for $f \in C_c^0(\mathbb{R}^n)$, and let Lebesgue measure be the associated positive, regular, Borel measure. With this description of Lebesgue measure, as opposed to the more tangible (but also more awkward) Lebesgue outer measure, we must verify that all the expected properties do hold.
[2.1] **Corollary:** Let $\mu$ be Lebesgue measure, induced by the Riesz-Markov-Kakutani theorem from the Riemann integral on $C^0_c(\mathbb{R}^n)$.

- $\mu$ is *translation-invariant* in the sense that $\mu(E + x) = \mu(E)$ for all $x \in \mathbb{R}^n$.
- The Lebesgue measure of a cube $(a_1, b_1) \times \ldots \times (a_n, b_n)$ is the product $\prod_i |b_i - a_i|$, and similarly for closed and half-open intervals and their products.

**Proof:** (Standard... [... iou ...])

///