10. Banach spaces $C^k[a, b]$

Paul Garrett  garrett@math.umn.edu  http://www.math.umn.edu/~garrett/

[This document is http://www.math.umn.edu/~garrett/m/real/notes_2017-18/10_spaces_Ck.pdf]

1. Banach spaces $C^k[a, b]$
2. Non-Banach limit $C^\infty[a, b]$ of Banach spaces $C^k[a, b]$

We specify natural topologies, in which differentiation or other natural operators are continuous, and so that the space is complete.

Many familiar and useful spaces of continuous or differentiable functions, such as $C[0, 1]$ giving Banach spaces.

complete metric structures, and are complete. In these cases, the metric $d(.)$ comes from a norm $|\cdot|$, on the functions, giving Banach spaces.

Other natural function spaces, such as $C^\infty[a, b]$, are not Banach, but still do have a metric topology and are complete: these are Fréchet spaces, appearing as (projective) limits of Banach spaces, as below. These lack some of the conveniences of Banach spaces, but their expressions as limits of Banach spaces is often sufficient.

1. Banach spaces $C^k[a, b]$

We give the vector space $C^k[a, b]$ of $k$-times continuously differentiable functions on an interval $[a, b]$ a metric which makes it complete. Mere pointwise limits of continuous functions easily fail to be continuous. First recall the standard

[1.1] Claim: The set $C^\infty(K)$ of complex-valued continuous functions on a compact set $K$ is complete with the metric $|f - g|_{C^\infty}$, with the $C^\infty$-norm $|f|_{C^\infty} = \sup_{x \in K} |f(x)|$.

Proof: This is a typical three-epsilon argument. To show that a Cauchy sequence $\{f_i\}$ of continuous functions has a pointwise limit which is a continuous function, first argue that $f_i$ has a pointwise limit at every $x \in K$. Given $\varepsilon > 0$, choose $N$ large enough such that $|f_i - f_j| < \varepsilon$ for all $i, j \geq N$. Then $|f_i(x) - f_j(x)| < \varepsilon$ for any $x \in K$. Thus, the sequence of values $f_i(x)$ is a Cauchy sequence of complex numbers, so has a limit $f(x)$. Further, given $\varepsilon' > 0$ choose $j \geq N$ sufficiently large such that $|f_j(x) - f(x)| < \varepsilon'$. For $i \geq N$

$$|f_i(x) - f(x)| \leq |f_i(x) - f_j(x)| + |f_j(x) - f(x)| < \varepsilon + \varepsilon'$$

This is true for every positive $\varepsilon'$, so $|f_i(x) - f(x)| \leq \varepsilon$ for every $x \in K$. That is, the pointwise limit is approached uniformly in $x \in [a, b]$.

To prove that $f(x)$ is continuous, for $\varepsilon > 0$, take $N$ be large enough so that $|f_i - f_j| < \varepsilon$ for all $i, j \geq N$. From the previous paragraph $|f_i(x) - f(x)| \leq \varepsilon$ for every $x$ and for $i \geq N$. Fix $i \geq N$ and $x \in K$, and choose a small enough neighborhood $U$ of $x$ such that $|f_i(x) - f_i(y)| < \varepsilon$ for any $y \in U$. Then

$$|f(x) - f(y)| \leq |f(x) - f_i(x)| + |f_i(x) - f_i(y)| + |f(y) - f_i(y)| \leq \varepsilon + |f_i(x) - f_i(y)| + \varepsilon < \varepsilon + \varepsilon + \varepsilon$$

Thus, the pointwise limit $f$ is continuous at every $x \in U$.  

Unsurprisingly, but significantly:

[1.2] Claim: For $x \in [a, b]$, the evaluation map $f \to f(x)$ is a continuous linear functional on $C^\infty[a, b]$.

Proof: For $|f - g|_{C^\infty} < \varepsilon$, we have

$$|f(x) - g(x)| \leq |f - g|_{C^\infty} < \varepsilon$$

1
Theorem: The normed metric space $\operatorname{d}erivative \lim$

Proof: For a Cauchy sequence $C$, differentiable
As usual, a real-valued or complex-valued function $f$ on a closed interval $[a, b] \subset \mathbb{R}$ is continuously differentiable when it has a derivative which is itself a continuous function. That is, the limit

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

exists for all $x \in [a, b]$, and the function $f'(x)$ is in $C^0[a, b]$. Let $C^k[a, b]$ be the collection of $k$-times continuously differentiable functions on $[a, b]$, with the $C^k$-norm

$$|f|_{C^k} = \sum_{0 \leq i \leq k} \sup_{x \in [a, b]} |f^{(i)}(x)| = \sum_{0 \leq i \leq k} |f^{(i)}|_{\infty}$$

where $f^{(i)}$ is the $i^{th}$ derivative of $f$. The associated metric on $C^k[a, b]$ is $|f - g|_{C^k}$.

Similar to the assertion about evaluation on $C^0[a, b]$,

[1.3] Claim: For $x \in [a, b]$ and $0 \leq j \leq k$, the evaluation map $f \to f^{(j)}(x)$ is a continuous linear functional on $C^k[a, b]$.

Proof: For $|f - g|_{C^k} < \varepsilon$,

$$|f^{(j)}(x) - g^{(j)}(x)| \leq |f - g|_{C^k} < \varepsilon$$

proving the continuity.  

We see that $C^k[a, b]$ is a Banach space:

[1.4] Theorem: The normed metric space $C^k[a, b]$ is complete.

Proof: For a Cauchy sequence $\{f_i\}$ in $C^k[a, b]$, all the pointwise limits $\lim_i f^{(j)}_i(x)$ of $j$-fold derivatives exist for $0 \leq j \leq k$, and are uniformly continuous. The issue is to show that $\lim_i f^{(j)}_i(x)$ is differentiable, with derivative $\lim_i f^{(j+1)}_i(x)$. It suffices to show that, for a Cauchy sequence $f_n$ in $C^1[a, b]$, with pointwise limits $f(x) = \lim_n f_n(x)$ and $g(x) = \lim_n f'_n(x)$ we have $g = f'$. By the fundamental theorem of calculus, for any index $i$,

$$f_i(x) - f_i(a) = \int_a^x f'_i(t) \, dt$$

Since the $f'_i$ uniformly approach $g$, given $\varepsilon > 0$ there is $i_o$ such that $|f'_i(t) - g(t)| < \varepsilon$ for $i \geq i_o$ and for all $t$ in the interval, so for such $i$

$$\left| \int_a^x f'_i(t) \, dt - \int_a^x g(t) \, dt \right| \leq \int_a^x |f'_i(t) - g(t)| \, dt \leq \varepsilon \cdot |x - a| \to 0$$

Thus,

$$\lim_i f_i(x) - f_i(a) = \lim_i \int_a^x f'_i(t) \, dt = \int_a^x g(t) \, dt$$

from which $f' = g$.  

By design, we have

[1.5] Theorem: The map $\frac{d}{dx} : C^k[a, b] \to C^{k-1}[a, b]$ is continuous.

Proof: As usual, for a linear map $T : V \to W$, by linearity $T v - T v' = T(v - v')$ it suffices to check continuity at 0. For Banach spaces the homogeneity $|\sigma \cdot v|_V = |\sigma| \cdot |v|_V$ shows that continuity is equivalent to existence of a constant $B$ such that $|Tv|_W \leq B \cdot |v|_V$ for $v \in V$. Then

$$\left| \frac{d}{dx} f \right|_{C^{k-1}} = \sum_{0 \leq i \leq k-1} \sup_{x \in [a, b]} \left| \frac{d^i}{dx^i} f \right| = \sum_{1 \leq i \leq k} \sup_{x \in [a, b]} |f^{(i)}(x)| \leq 1 \cdot |f|_{C^k}$$

proving the continuity.
2. Non-Banach limit \( C^\infty[a, b] \) of Banach spaces \( C^k[a, b] \)

The space \( C^\infty[a, b] \) of infinitely differentiable complex-valued functions on a (finite) interval \([a, b]\) in \( \mathbb{R} \) is not a Banach space. \(^1\) Nevertheless, the topology is *completely determined* by its relation to the Banach spaces \( C^k[a, b] \). That is, there is a *unique* reasonable topology on \( C^\infty[a, b] \). After explaining and proving this uniqueness, we also show that this topology is *complete metric*.

This function space can be presented as

\[
C^\infty[a, b] = \bigcap_{k \geq 0} C^k[a, b]
\]

and we reasonably require that whatever topology \( C^\infty[a, b] \) should have, each inclusion \( C^\infty[a, b] \rightarrow C^k[a, b] \) is continuous.

At the same time, given a family of *continuous linear* maps \( Z \rightarrow C^k[a, b] \) from a vector space \( Z \) in some reasonable class, with the *compatibility* condition of giving commutative diagrams

\[
\begin{array}{ccc}
C^k[a, b] & \rightarrow & C^{k-1}[a, b] \\
\downarrow & & \downarrow \\
Z & \rightarrow & Z
\end{array}
\]

the image of \( Z \) actually lies in the intersection \( C^\infty[a, b] \). Thus, diagrammatically, for every family of compatible maps \( Z \rightarrow C^k[a, b] \), there is a *unique* \( Z \rightarrow C^\infty[a, b] \) fitting into a commutative diagram

\[
\begin{array}{cccc}
C^\infty[a, b] & \rightarrow & \cdots & \rightarrow C^1[a, b] & \rightarrow C^0[a, b] \\
\downarrow & & & & \downarrow \\
Z & \rightarrow & \cdots & \rightarrow Z
\end{array}
\]

We require that this induced map \( Z \rightarrow C^\infty[a, b] \) is *continuous*.

When we know that these conditions are met, we would say that \( C^\infty[a, b] \) is the (projective) *limit* of the spaces \( C^k[a, b] \), written

\[
C^\infty[a, b] = \lim_k C^k[a, b]
\]

with implicit reference to the inclusions \( C^{k+1}[a, b] \rightarrow C^k[a, b] \) and \( C^\infty[a, b] \rightarrow C^k[a, b] \).

**[2.1] Claim:** Up to unique isomorphism, there exists at most one topology on \( C^\infty[a, b] \) such that to every compatible family of continuous linear maps \( Z \rightarrow C^k[a, b] \) from a topological vector space \( Z \) there is a unique continuous linear \( Z \rightarrow C^\infty[a, b] \) fitting into a commutative diagram as just above.

**Proof:** Let \( X, Y \) be \( C^\infty[a, b] \) with two topologies fitting into such diagrams, and show \( X \approx Y \), and for a unique isomorphism. First, claim that the identity map \( \text{id}_X : X \rightarrow X \) is the only map \( \varphi : X \rightarrow X \) fitting into a commutative diagram

\[\text{[1]}\] \footnote{It is not essential to prove that there is no reasonable Banach space structure on \( C^\infty[a, b] \), but this can be readily proven in a suitable context.}
Indeed, given a compatible family of maps $X \to C^k[a,b]$, there is unique $\varphi$ fitting into

$$
\begin{array}{c}
X \\
\varphi \\
\end{array} 
\xrightarrow{\cdots} 
\begin{array}{c}
C^1[a,b] \\
\longrightarrow \\
C^\infty[a,b] \\
\end{array} 
\xrightarrow{\cdots} 
\begin{array}{c}
X \\
\varphi \\
\end{array}
$$

Since the identity map $\text{id}_X$ fits, necessarily $\varphi = \text{id}_X$. Similarly, given the compatible family of inclusions $Y \to C^k[a,b]$, there is unique $f : Y \to X$ fitting into

$$
\begin{array}{c}
X \\
\varphi \\
\end{array} 
\xrightarrow{\cdots} 
\begin{array}{c}
C^1[a,b] \\
\longrightarrow \\
C^\infty[a,b] \\
\end{array} 
\xrightarrow{\cdots} 
\begin{array}{c}
Y \\
\varphi \\
\end{array}
$$

Similarly, given the compatible family of inclusions $X \to C^k[a,b]$, there is unique $g : X \to Y$ fitting into

$$
\begin{array}{c}
Y \\
\varphi \\
\end{array} 
\xrightarrow{\cdots} 
\begin{array}{c}
C^1[a,b] \\
\longrightarrow \\
C^\infty[a,b] \\
\end{array} 
\xrightarrow{\cdots} 
\begin{array}{c}
X \\
\varphi \\
\end{array}
$$

Then $f \circ g : X \to X$ fits into a diagram

$$
\begin{array}{c}
X \\
\varphi \\
\end{array} 
\xrightarrow{\cdots} 
\begin{array}{c}
C^1[a,b] \\
\longrightarrow \\
C^\infty[a,b] \\
\end{array} 
\xrightarrow{\cdots} 
\begin{array}{c}
X \\
\varphi \\
\end{array}
$$

Therefore, $f \circ g = \text{id}_X$. Similarly, $g \circ f = \text{id}_Y$. That is, $f, g$ are mutual inverses, so are isomorphisms of topological vector spaces. ///

Existence of a topology on $C^\infty[a,b]$ satisfying the condition above will be proven by identifying $C^\infty[a,b]$ as the obvious diagonal closed subspace of the topological product of the limitands $C^k[a,b]$: $C^\infty[a,b] = \{ \{ f_k : f_k \in C^k[a,b] \} : f_k = f_{k+1} \text{ for all } k \}$

An arbitrary product of topological spaces $X_\alpha$ for $\alpha$ in an index set $A$ is a topological space $X$ with (projections) $p_\alpha : X \to X_\alpha$, such that every family $f_\alpha : Z \to X_\alpha$ of maps from any other topological space $Z$ factors through the $p_\alpha$ uniquely, in the sense that there is a unique $f : Z \to X$ such that $f_\alpha = p_\alpha \circ f$ for all $\alpha$. Pictorially, all triangles commute in the diagram

$$
\begin{array}{c}
Z \\
\xrightarrow{f} \\
X \\
\xrightarrow{p_\beta} \\
\cdots X_\alpha \\
\xrightarrow{f_\alpha} \\
\cdots X_{\beta} \\
\xrightarrow{p_\beta} \\
\cdots \\
\end{array}
$$
A similar argument to that for uniqueness of limits proves *uniqueness* of products up to unique isomorphism. *Construction* of products is by putting the usual product topology with basis consisting of products $\prod_i Y_i$ with $Y_i = X_i$ for all but finitely-many indices, on the Cartesian product of the sets $X_i$, whose existence we grant ourselves. Proof that this usual is a product amounts to unwinding the definitions. By uniqueness, in particular, despite the plausibility of the *box topology* on the product, it cannot function as a product topology since it differs from the standard product topology in general.

[2.2] **Claim:** Giving the diagonal copy of $C^\infty[a,b]$ inside $\prod_k C^k[a,b]$ the subspace topology yields a (projective) limit topology.

**Proof:** The projection maps $p_k : \prod_j C^j[a,b] \to C^k[a,b]$ from the whole product to the factors $C^k[a,b]$ are continuous, so their restrictions to the diagonally imbedded $C^\infty[a,b]$ are continuous. Further, letting $i_k : C^k[a,b] \to C^{k-1}[a,b]$ be the inclusion, on that diagonal copy of $C^\infty[a,b]$ we have $i_k \circ p_k = p_{k-1}$ as required.

On the other hand, any family of maps $\varphi_k : Z \to C^k[a,b]$ induces a map $\tilde{\varphi} : Z \to \prod C^k[a,b]$ such that $p_k \circ \tilde{\varphi} = \varphi_k$, by the property of the product. *Compatibility* $i_k \circ \varphi_k = \varphi_{k-1}$ implies that the image of $\tilde{\varphi}$ is inside the diagonal, that is, inside the copy of $C^\infty[a,b]$. \[\]

A countable product of metric spaces $X_k$ with metrics $d_k$ has no canonical single metric, but is metrizable. One of many topologically equivalent metrics is the usual

$$d(\{x_k\}, \{y_k\}) = \sum_{k=0}^{\infty} 2^{-k} \frac{d_k(x_k - y_k)}{d_k(x_k - y_k) + 1}$$

When the metric spaces $X_k$ are complete, the product is complete. A closed subspace of a complete metrizable space is complete metrizable, so we have

[2.3] **Corollary:** $C^\infty[a,b]$ is complete metrizable. \[\]

Abstracting the above, for a (not necessarily countable) family

$$\ldots \xrightarrow{\varphi_2} B_1 \xrightarrow{\varphi_1} B_\alpha$$

of Banach spaces with continuous linear transition maps as indicated, not necessarily requiring the continuous linear maps to be injective (or surjective), a (projective) limit $\lim_i B_i$ is a topological vector space with continuous linear maps $\lim_i B_i \to B_j$ such that, for every compatible family of continuous linear maps $Z \to B_i$ there is unique continuous linear $Z \to \lim_i B_i$ fitting into

$$\lim_i B_i \xrightarrow{\varphi_2} B_1 \xrightarrow{\varphi_1} B_\alpha$$

The same uniqueness proof as above shows that there is at most one topological vector space $\lim_i B_i$. For *existence* by *construction*, the earlier argument needs only minor adjustment. The conclusion of complete metrizability would hold when the family is countable.

Before declaring $C^\infty[a,b]$ to be a *Fréchet* space, we must certify that it is locally convex, in the sense that every point has a local basis of convex opens. Normed spaces are immediately locally convex, because open balls are convex: for $0 \leq t \leq 1$ and $x,y$ in the $\varepsilon$-ball at $0$ in a normed space,

$$|tx + (1-t)y| \leq |tx| + |(1-t)y| \leq t|x| + (1-t)|y| < t \cdot \varepsilon + (1-t) \cdot \varepsilon = \varepsilon$$
Product topologies of locally convex vectorspaces are locally convex, from the construction of the product. The construction of the limit as the diagonal in the product, with the subspace topology, shows that it is locally convex. In particular, countable limits of Banach spaces are locally convex, hence, are Fréchet. All spaces of practical interest are locally convex for simple reasons, so demonstrating local convexity is rarely interesting.

[2.4] Theorem: \( \frac{d}{dx} : C^\infty[a, b] \to C^\infty[a, b] \) is continuous.

Proof: In fact, the differentiation operator is characterized via the expression of \( C^\infty[a, b] \) as a limit. We already know that differentiation \( d/dx \) gives a continuous map \( C^k[a, b] \to C^{k-1}[a, b] \). Differentiation is compatible with the inclusions among the \( C^k[a, b] \). Thus, we have a commutative diagram

\[
\begin{array}{cccccc}
C^\infty[a, b] & \to & C^k[a, b] & \to & C^{k-1}[a, b] & \to & \ldots \\
\downarrow & & \uparrow & & \uparrow & & \\
C^\infty[a, b] & \to & C^k[a, b] & \to & C^{k-1}[a, b] & \to & \ldots \\
\end{array}
\]

Composing the projections with \( d/dx \) gives (dashed) induced maps from \( C^\infty[a, b] \) to the limitands, inducing a unique (dotted) continuous linear map to the limit, as in

\[
\begin{array}{cccccc}
C^\infty[a, b] & \to & C^k[a, b] & \to & C^{k-1}[a, b] & \to & \ldots \\
\downarrow & & \uparrow & & \uparrow & & \\
C^\infty[a, b] & \to & C^k[a, b] & \to & C^{k-1}[a, b] & \to & \ldots \\
\end{array}
\]

This proves the continuity of differentiation in the limit topology.

In a slightly different vein, we have

[2.5] Claim: For all \( x \in [a, b] \) and for all non-negative integers \( k \), the evaluation map \( f \to f^{(k)}(x) \) is a continuous linear map \( C^\infty[a, b] \to \mathbb{C} \).

Proof: The inclusion \( C^\infty[a, b] \to C^k[a, b] \) is continuous, and the evaluation of the \( k^{th} \) derivative is continuous.

///