04b. Product measures and Fubini-Tonelli theorem

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1. Product measures
2. Fubini-Tonelli theorem(s)
3. Completions of measures

1. Product measures, completions of measures

Let \( X, \mu \) and \( Y, \nu \) be measure spaces with corresponding \( \sigma \)-algebras \( A, B \). Assume \( X \) and \( Y \) are \( \sigma \)-finite, in the sense that they are countable unions of finite-measure sets.

First, the product \( \sigma \)-algebra is the \( \sigma \)-algebra in \( X \times Y \) generated by all products \( E \times F \) with \( E \in A \) and \( F \in B \).

For iterated integrals to make sense, we need to check a few things. For \( E \in A \times B \), for \( x \in X \) and \( y \in Y \), let
\[
E_x = \{ y \in Y : (x, y) \in E \} \quad \text{and} \quad E^y = \{ x \in X : (x, y) \in E \}
\]
As a consistency check, we have

[1.1] Theorem: For \( E \in A \times B \), for \( x \in X \) and \( y \in Y \), \( E_x \in A \) and \( E^y \in B \). The function \( x \to \nu(E_x) \) is \( \mu \)-measurable, \( y \to \mu(E^y) \) is \( \nu \)-measurable, and
\[
\int_X \nu(E_x) \, d\mu(x) = \int_Y \mu(E^y) \, d\nu(y)
\]

Proof: [... iou ...] // /

Then the product measure \( \mu \times \nu \) can be defined in the expected fashion: for \( E \in A \times B \),
\[
(\mu \times \nu)(E) = \int_X \nu(E_x) \, d\mu(x) = \int_Y \mu(E^y) \, d\nu(y)
\]

2. Fubini-Tonelli theorem(s)

Let \( X, \mu \) and \( Y, \nu \) be measure spaces with corresponding \( \sigma \)-algebras \( A, B \). Assume \( X \) and \( Y \) are \( \sigma \)-finite.

[2.1] Theorem: (Fubini-Tonelli) For complex-valued measurable \( f, g \), if any one of
\[
\int_X \int_Y |f(x, y)| \, d\mu(x) \, d\nu(y) \quad \int_Y \int_X |f(x, y)| \, d\nu(y) \, d\mu(x) \quad \int_{X \times Y} |f(x, y)| \, d\mu \times \nu
\]
is finite, then they all are finite, and are equal. For \([0, +\infty] \)-valued functions \( f \),
\[
\int_X \int_Y f(x, y) \, d\mu(x) \, d\nu(y) = \int_Y \int_X f(x, y) \, d\nu(y) \, d\mu(x) = \int_{X \times Y} f(x, y) \, d\mu \times \nu
\]
although the values may be \(+\infty\).
Proof: [...] iou [...]

To explain what the product measure $\mu \times \nu$ should be, and also for a proof of the theorem, we need the notion of monotone class. A monotone class in a set $X$ is a set $\mathcal{M}$ of subsets of $X$ closed under countable ascending unions and under countable descending intersections. That is, if

$$M_1 \subset M_2 \subset M_3 \subset \ldots$$

$$N_1 \supset N_2 \supset N_3 \supset \ldots$$

are collections of sets in $\mathcal{M}$, then

$$\bigcup_i M_i \quad \bigcap_i N_i$$

both lie in $\mathcal{M}$, as well. Another characterization of $A \times B$ is that it is the smallest monotone class containing all products $E \times F$ with $E \in A$ and $F \in B$.

Let $f$ be a $A \times B$-measurable function on $X \times Y$. (Note that this does not depend upon having a ‘product measure’, but only upon the sigma-algebra!) Then all the functions

$$x \to f(x, y) \quad \text{(for fixed } y \in Y)$$

$$y \to f(x, y) \quad \text{(for fixed } x \in X)$$

are measurable (in appropriate senses). In particular, we could apply this to the characteristic function of a set $G \in A \times B$.

Now we come to the point where the sigma-finiteness of $X$ and $Y$ is necessary. For $G \in A \times B$, let

$$f(x) = \nu(G_x) \quad g(y) = \mu(G_y)$$

where $G_x, G_y$ are as above. We have already noted that $f, g$ are measurable. Further,

$$\int_X f(x) \, d\mu(x) = \int_Y g(y) \, d\nu(y)$$

This is proven by showing that the collection of $G$ for which the conclusion is true is a monotone class containing all products $E \times F$.

In light of the latter equality, we can define the product measure $\mu \times \nu$ on $G \in A \times B$ by

$$(\mu \times \nu)(G) = \int_X f(x) \, d\mu(x) = \int_Y g(y) \, d\nu(y)$$

with notation as just above. The countable additivity follows from a preliminary version of Fubini’s theorem, namely that if $f_i$ are countably-many $[0, +\infty]$-valued functions on a measure space $\Omega$, then

$$\int_\Omega \sum_i f_i = \sum_i \int_\Omega f_i$$

which itself is a little corollary of the monotone convergence theorem.

sectionCompletions of measures

For example, a reasonable measure on $\mathbb{R}^m \times \mathbb{R}^n$ should include many sets not expressible as countable unions of products $E \times F$ where $E \subset \mathbb{R}^m$ and $F \subset \mathbb{R}^n$. For example, diagonal subsets of the form

$$D = \{(x, x) : 0 \leq x \leq 1\} \subset \mathbb{R}^2$$

are not countable unions of products, but should surely be measurable.

One way to accomplish this is by completion of the product measure.
Then the completion of $\mu \times \nu$ further assigns measure 0 to any subset $S$ of $T \in A \times B$ with $(\mu \times \nu)(T) = 0$, and adjoins all such sets to the $\sigma$-algebra $A \times B$.

[2.2] Claim: Lebesgue measure on $\mathbb{R}^m \times \mathbb{R}^n$ is the completion of the product of Lebesgue measures on $\mathbb{R}^m$ and $\mathbb{R}^n$.

Proof: [... iou ...] ///

Completing a product measure is usually what we want, but it slightly complicates the statement of the corresponding Fubini-Tonelli theorem:

[2.3] Theorem: Let $X, A, \mu$ and $Y, B, \nu$ be complete measure spaces, with $X, Y$ $\sigma$-finite. Let $f$ be a function on $X \times Y$ measurable with respect to the completion of the product measure. Then $x \to f(x, y)$ and $y \to f(x, y)$ are $\mu$-measurable and $\nu$-measurable (only) almost everywhere.

Proof: [... iou ...] ///

[2.4] Remark: To be precise, completeness is a property of $\sigma$-algebras, not of measures.