1. **Examples: spaces $L^p$**

Given a measure space $X$, for $1 \leq p < \infty$ the usual $L^p$ spaces are

$$L^p(X) = \{\text{measurable } f : |f|_{L^p} < \infty\} \text{ modulo } \sim$$

with the usual $L^p$ norm

$$|f|_{L^p} = \left( \int_X |f|^p \right)^{1/p}$$

and associated metric

$$d(f, g) = |f - g|_{L^p}$$

taking the quotient by the equivalence relation

$$f \sim g \text{ if } f - g = 0 \text{ off a set of measure 0}$$

**[1.1] Remark:** These $L^p$ functions have inevitably ambiguous pointwise values, in conflict with the naive formal definition of function.

A simple instance of this construction is

$$\ell^p = \{\text{complex sequences } \{c_i\} \text{ with } \sum_i |c_i|^p < \infty\}$$

with norm $|(c_1, c_2, \ldots)|_{\ell^p} = (\sum_i |c_i|^p)^{1/p}$. The analogue of the following theorem for $\ell^p$ is more elementary.

**[1.2] Theorem:** The space $L^p(X)$ is a complete metric space.

**[1.3] Remark:** In fact, as used in the proof, a Cauchy sequence $f_i$ in $L^p(X)$ has a subsequence converging pointwise off a set of measure 0 in $X$.

**Proof:** The triangle inequality here is Minkowski’s inequality. To prove completeness, choose a subsequence $f_{n_i}$ such that

$$|f_{n_i} - f_{n_{i+1}}|_p < 2^{-i}$$

and put

$$g_n(x) = \sum_{1 \leq i \leq n} |f_{n_{i+1}}(x) - f_{n_i}(x)|$$

and

$$g(x) = \sum_{1 \leq i < \infty} |f_{n_{i+1}}(x) - f_{n_i}(x)|$$

The infinite sum is not necessarily claimed to converge to a finite value for every $x$. The triangle inequality shows that $|g_n|_p \leq 1$. Fatou’s Lemma asserts that for $[0, \infty]$-valued measurable functions $h_i$

$$\int_X \left( \lim \inf_i h_i \right) \leq \lim \inf_i \int_X h_i$$
Thus, \(|g|_p \leq 1\), so is finite. Thus,

\[ f_{n_1}(x) + \sum_{i \geq 1} (f_{n_{i+1}}(x) - f_{n_i}(x)) \]

converges for almost all \(x \in X\). Let \(f(x)\) be the sum at points \(x\) where the series converges, and on the measure-zero set where the series does not converge put \(f(x) = 0\). Certainly

\[ f(x) = \lim_i f_n(x) \quad \text{(for almost all } x) \]

Now prove that this almost-everywhere pointwise limit is the \(L^p\)-limit of the original sequence. For \(\varepsilon > 0\) take \(N\) such that \(|f_m - f_n|_p < \varepsilon\) for \(m, n \geq N\). Fatou’s lemma gives

\[ \int |f - f_n|^p \leq \liminf_i \int |f_n - f|^p \leq \varepsilon^p \]

Thus \(f - f_n\) is in \(L^p\) and hence \(f\) is in \(L^p\). And \(|f - f_n|_p \rightarrow 0\). ///

[1.4] Theorem: For a locally compact Hausdorff topological space \(X\) with positive regular Borel measure \(\mu\), the space \(\mathcal{C}_c^0(X)\) of compactly-supported continuous functions is dense in \(L^p(X, \mu)\).

**Proof:** From the definition of integral attached to a measure, an \(L^p\) function is approximable in \(L^p\) metric by a simple function, that is, a measurable function assuming only finitely-many values. That is, a simple function is a finite linear combination of characteristic functions of measurable sets \(E\). Thus, it suffices to approximate characteristic functions of measurable sets by continuous functions. The assumed regularity of the measure gives compact \(K\) and open \(U\) such that \(K \subset E \subset U\) and \(\mu(U - E) < \varepsilon\), for given \(\varepsilon > 0\). Urysohn’s lemma says that there is continuous \(f\) identically 1 on \(K\) and identically 0 off \(U\). Thus, \(f\) approximates the characteristic function of \(E\). ///

[1.5] Corollary: For locally compact Hausdorff \(X\) with regular Borel measure \(\mu\), \(L^p(X, \mu)\) is the \(L^p\)-metric completion of \(\mathcal{C}_c^0(X)\), the compactly-supported continuous functions. ///

[1.6] Remark: Defining \(L^p(X, \mu)\) to be the \(L^p\) completion of \(\mathcal{C}_c^0(X)\) avoids discussion of ambiguous values on sets of measure zero.

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### 2. Convexity and inequalities

A function \(f\) on an interval \((a, b) \subset \mathbb{R}\) is convex when its graph bends upward, in the sense that a line segment connecting two points on the graph lies above the graph. That is,

\[ f(tx + (1-t)y) \geq tf(x) + (1-t)f(y) \quad \text{(for } 0 \leq t \leq 1 \text{ and } a < x < y < b) \]

The prototype is the exponential function \(x \rightarrow e^x\).

[2.1] Claim: Convex \(\mathbb{R}\)-valued functions on an open interval \((a, b)\) (allowing \(a = -\infty\) and/or \(b = +\infty\)) are continuous.

**Proof:** Let \(g\) be continuous on \((a, b)\) and take \(x \in (a, b)\). Fix any \(s, t\) such that \(a < s < x < t < b\). For \(y\) in the range \(x < y < t\), the point \((y, g(y))\) is on or above the line through \((s, g(s))\) and \((x, g(x))\), and is below the line through \((x, g(x))\) and \((t, g(t))\), so \(g(y) \rightarrow g(x)\) as \(y \rightarrow x^+\). For \(s < y < x\), the same argument gives left-continuity. ///

[2.2] Theorem: (Jensen’s inequality) Let \(X\) be a measure space with positive measure of total measure 1. Let \(f \in L^1(X)\) be an \(\mathbb{R}\)-valued function on \(X\) with \(a < f(x) < b\) for all \(x \in X\), where \(a, b\) can also be \(-\infty\) and \(+\infty\). For convex \(g\) on \((a, b)\),

\[ g \left( \int_X f \right) \leq \int_X g \cdot f \]
**Proof:** First, \( a < f(x) < b \) gives \( a < \int_X f < b \). The convexity condition can be rewritten as the condition that slopes of secants increase from left to right. Thus, for example,

\[
\frac{g(y) - g(x)}{y - x} \leq \frac{g(z) - g(y)}{z - y} \quad \text{(for } x < y < z \text{ inside } (a, b))
\]

Applying this with \( y = \int_X f \),

\[
\frac{g(\int_X f) - g(x)}{\int_X f - x} \leq \frac{g(z) - g(\int_X f)}{z - \int_X f} \quad \text{(for all } a < x < \int_X f \text{ and for all } \int_X f < z < b)
\]

With

\[
S = \sup_{a < x < \int_X f} \frac{g(\int_X f) - g(x)}{\int_X f - x}
\]

we have

\[
\frac{g(\int_X f) - g(x)}{\int_X f - x} \leq S \leq \frac{g(z) - g(\int_X f)}{z - \int_X f} \quad \text{(for all } a < x < \int_X f \text{ and for all } \int_X f < z < b)
\]

Thus, from the left half of the latter inequality,

\[
g(x) \geq g(\int_X f) + S \cdot (x - \int_X f) \quad \text{(for } a < x < \int_X f)\]

and from the right half

\[
g(z) \geq g(\int_X f) + S \cdot (z - \int_X f) \quad \text{(for } \int_X f < z < b)\]

Thus,

\[
g(w) \geq g(\int_X f) + S \cdot (w - \int_X f) \quad \text{(for all } w \text{ in the range } a < w < b)\]

In particular, letting \( w = f(x) \) now with \( x \in X \),

\[
g(f(x)) \geq g(\int_X f) + S \cdot (f(x) - \int_X f) \quad \text{(for all } w \text{ in the range } a < w < b)\]

Since the convex function \( g \) is continuous, \( g \circ f \) is measurable. Integrating in \( x \in X \), using the fact that the total measure is 1,

\[
\int_X g \circ f \geq g(\int_X f) + S \cdot (\int_X f - \int_X f) = g(\int_X f) + S \cdot 0
\]

as claimed. ///

**[2.3 Corollary]** *(Arithmetic-geometric mean inequality)* For positive real numbers \( a_1, \ldots, a_n \),

\[
(a_1 a_2 \ldots a_n)^{1/n} \leq \frac{a_1 + a_2 + \ldots + a_n}{n}
\]

**Proof:** In Jensen’s inequality, take \( g(x) = e^x \), take \( X \) a finite set with \( n \) (distinct) elements \( \{x_1, \ldots, x_n\} \), with each point having measure \( 1/n \), and \( f(x_i) = \log a_i \). Jensen’s inequality gives

\[
\exp \left( \frac{\log a_1 + \ldots + \log a_n}{n} \right) \leq \frac{e^{\log a_1} + \ldots + e^{\log a_n}}{n}
\]
which gives the assertion.

Conjugate exponents are numbers $p, q > 1$ such that

$$\frac{1}{p} + \frac{1}{q} = 1$$

For example, $p$ and $\frac{p}{p-1}$ are conjugate exponents.

Generalizing the Cauchy-Schwarz-Bunyakowsky inequality,

[2.4] Corollary: (Hölder) For conjugate exponents $p, q$ and $[0, +\infty]$-valued measurable functions $f, g$,

$$\int_X f \cdot g \leq \left( \int_X f^p \right)^{\frac{1}{p}} \cdot \left( \int_X g^q \right)^{\frac{1}{q}}$$

Proof: The assertion is trivial if either integral on the right-hand side is $+\infty$ or 0, so suppose the two quantities

$$I = \left( \int_X f^p \right)^{\frac{1}{p}} \quad J = \left( \int_X g^q \right)^{\frac{1}{q}}$$

are finite and non-zero. Renormalize by taking $\varphi = f/I$ and $\psi = g/J$, so that $\int \varphi^p = 1 = \int \psi^q$. For $x \in X$ with $0 < \varphi(x) < \infty$ and $0 < \psi(x) < \infty$, there are real numbers $u, v$ such that $e^{u/p} = \varphi(x)$ and $e^{v/q} = \psi(x)$.

Invoking Jensen’s inequality on a measure space with just two points with measures $\frac{1}{p}$ and $\frac{1}{q}$, using the convexity of the exponential function,

$$\varphi(x) \psi(x) = e^{\frac{u}{p} + \frac{v}{q}} \leq e^{\frac{u}{p}} + e^{\frac{v}{q}} = \frac{\varphi(x)^p}{p} + \frac{\psi(x)^q}{q}$$

Integrating,

$$\int_X \varphi \cdot \psi \leq \int_X \frac{\varphi(x)^p}{p} + \frac{\psi(x)^q}{q} = \frac{1}{p} + \frac{1}{q} = 1$$

From the renormalization, we are done.

For the triangle inequality in $L^p$ spaces for general $p$, we need

[2.5] Corollary: (Minkowski) For $1 < p < +\infty$ and $[0, +\infty]$-valued measurable functions $f, g$,

$$\left( \int_X (f + g)^p \right)^{\frac{1}{p}} \leq \left( \int_X f^p \right)^{\frac{1}{p}} + \left( \int_X g^p \right)^{\frac{1}{p}}$$

Proof: We prove Minkowski’s inequality from Hölder’s, using the conjugate exponents $p$ and $q = \frac{p}{p-1}$.

$$\int (f + g)^p = \int f \cdot (f + g)^{p-1} + \int g \cdot (f + g)^{p-1}$$

$$\leq \left( \int f^p \right)^{\frac{1}{p}} \cdot \left( \int (f + g)^{(p-1)q} \right)^{\frac{1}{q}} + \left( \int g^p \right)^{\frac{1}{p}} \cdot \left( \int (f + g)^{(p-1)q} \right)^{\frac{1}{q}}$$

$$= \left[ \left( \int f^p \right)^{\frac{1}{p}} + \left( \int g^p \right)^{\frac{1}{p}} \right] \cdot \left( \int (f + g)^p \right)^{\frac{p-1}{p}}$$

Dividing through by $\left( \int (f + g)^p \right)^{\frac{p-1}{p}}$ gives Minkowski’s inequality.