1. Convolution on $L^1(\mathbb{R}^n)$

The formulaic definition of convolution of $f, g \in L^1(\mathbb{R}^n)$ is as a pointwise (a.e.) function

$$(f \ast g)(x) = \int_{\mathbb{R}^n} f(x-y) g(y) \, dy$$

For each fixed $x$, it is not a priori clear that the integral converges. If $f, g \in L^2$, then Cauchy-Schwarz-Bunyakowsky could be invoked to show that the integral converges absolutely, but on $\mathbb{R}^n$ there are $L^1$ functions that are not $L^2$. So we need

[1.1] Claim: For $f, g \in L^1(\mathbb{R}^n)$, $f \ast g \in L^2(\mathbb{R}^n)$, and

$$|f \ast g|_{L^1} \leq |f|_{L^1} \cdot |g|_{L^1}$$

Proof: [... iou ...]

2. Structural meaning of convolution

The right translation action of $\mathbb{R}^n$ on functions on $\mathbb{R}^n$ is

$$(R_g f)(x) = f(x+g) \quad \text{for } x, g \in \mathbb{R}^n$$

The right invariance of the measure/integral immediately gives the invariance of the $L^2$ norm, for example:

$$|R_g f|_{L^2}^2 = \int_{\mathbb{R}^n} |g(x)|^2 \, dx = \int_{\mathbb{R}^n} |f(x)|^2 \, dx = |f|_{L^2}^2$$

[2.1] Claim: The map $\mathbb{R}^n \times L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ by $g \times f \to R_g f$ is continuous.

Proof: Fix $f \in L^2(\mathbb{R}^n)$, and take $\varepsilon > 0$. By density of $C_c^\infty(\mathbb{R}^n)$ in $L^2(\mathbb{R}^n)$, take $\varphi \in C_c^\infty(\mathbb{R}^n)$ such that $|f - \varphi|_{L^2} < \varepsilon$. Since $\varphi$ is compactly supported, $\varphi$ is uniformly continuous: for all $\varepsilon' > 0$, there is a neighborhood $N$ of $e \in \mathbb{R}^n$ such that $|\varphi(xh) - \varphi(x)| < \varepsilon'$ for all $h \in N$, for all $x \in \mathbb{R}^n$. For $g \in N$,

$$|R_g f - f|_{L^2} \leq |R_g f - R_g \varphi|_{L^2} + |R_g \varphi - \varphi|_{L^2} + |\varphi - f|_{L^2}$$
\begin{align*}
&\leq |f - \varphi|_{L^2} + \varepsilon' \cdot \text{meas} (\text{spt } \varphi) + |\varphi - f|_{L^2} = \varepsilon + \varepsilon' \cdot \text{meas} (\text{spt } \varphi) + \varepsilon
\end{align*}
Given \( \varepsilon \) and \( \varphi \), shrink \( N \) so that \( \varepsilon' \leq \text{meas} (\text{spt } \varphi) \), so \( |R_g f - f|_{L^2} < 3\varepsilon \) for \( g \in N \).

\section*{Integral-operator action of \( C_0^c(\mathbb{R}^n) \) on functions on \( \mathbb{R}^n \):}
Let \( \varphi \in C_0^c(\mathbb{R}^n) \) act on functions on \( \mathbb{R}^n \) by
\[
(\varphi \cdot f)(x) = \int_{\mathbb{R}}^n \varphi(g) \cdot f(x + g) \, dg
\]

**Convolution** We do not need to define convolution of \( C_0^c(\mathbb{R}^n) \) functions, but, rather, discover what kind of product on such functions is compatible with repeated application of the integral operators. That is, for \( \varphi, \psi \in C_0^c(\mathbb{R}^n) \), we want
\[
(\varphi * \psi) \cdot f = \varphi \cdot (\psi \cdot f)
\]

It hardly matters what topological vector space \( f \) lies in, whether or not it is a space of functions on \( \mathbb{R}^n \), since the same identity should hold regardless.

Compute directly, granting\(^\text{[1]}\) that continuous operators commute with these integrals and, of course, scalars commute with linear operators:
\[
\varphi \cdot (\psi \cdot f) = \int_{\mathbb{R}}^n \varphi(g) R_g \int_{\mathbb{R}}^n \psi(h) R_h f \, dh \, dg = \int_{\mathbb{R}}^n \varphi(g) \psi(h) R_g R_h f \, dh \, dg
\]
\[
= \int_{\mathbb{R}}^n \int_{\mathbb{R}}^n \varphi(g) \psi(h) R_{g+h} f \, dh \, dg
\]
At this point, there are two possible courses of action, either replace \( g \) by \( g - h \), or \( h \) by \( h - g \). Both choices are completely reasonable, but in the non-commutative case the appearances would be different. Let’s replace \( g \) by \( g - h \), assuming that \( dg \) refers to a right invariant measure on \( \mathbb{R}^n \). First interchanging the order of integration, do the change of variables, and then change back:
\[
\varphi \cdot (\psi \cdot f) = \int_{\mathbb{R}}^n \left( \int_{\mathbb{R}}^n \varphi(g-h) \psi(h) R_g f \, dg \right) dh = \int_{\mathbb{R}}^n \left( \int_{\mathbb{R}}^n \varphi(g-h) \psi(h) \, dh \right) R_g f \, dg
\]
\[
= \left( \int_{\mathbb{R}}^n \varphi(g-h) \psi(h) \, dh \right) \cdot f
\]
That is, we have more-or-less proven

\section*{[2.2] Proposition: Convolution}
\[
(\varphi * \psi)(g) = \int_{\mathbb{R}}^n \varphi(g-h) \psi(h) \, dh
\]
for \( \varphi, \psi \in L^1 \) gives the associativity
\[
(\varphi * \psi) \cdot f = \varphi \cdot (\psi \cdot f) \quad \text{(for all } f \in L^2(\mathbb{R}^n))\]

\section*{[2.3] Remark:}
In fact, the above discussion applies, with reasonable modifications, to arbitrary topological groups \( G \) in place of \( \mathbb{R}^n \), and very general topological vector spaces in place of \( L^2(\mathbb{R}^n) \).

\footnote{\text{[1]} Since the present argument does not show that continuous linear operators move inside the integrals, it remains a heuristic. But, eventually, we can rigorize it, in terms of Gelfand-Pettis vector-valued integrals.}
3. $\hat{f \ast g} = \hat{f} \ast \hat{g}$

The idea is that Fourier transform converts convolution to pointwise multiplication.

[3.1] Claim: For $f, g \in L^1(\mathbb{R}^n)$,

$$\hat{f \ast g} = \hat{f} \ast \hat{g}$$

Indeed, from above, $f \ast g \in L^1$, so the Fourier transform integral converges absolutely. Also, $\hat{f}$ and $\hat{g}$ are in fact continuous, by Riemann-Lebesgue, so there is certainly no ambiguity in talking about multiplication of point-wise values.

Proof: [...] iou ...]

4. $\delta$ as unit in convolution algebra

With suitable hypotheses on $f$,

[4.1] Claim: $\delta \ast f = f \ast \delta = f$ and $\delta' \ast f = f \ast \delta' = f'$.

[... iou ...]

5. Cautionary example

Disturbingly, associativity does not hold for arbitrary triples of distributions:

[5.1] Claim: Let $H$ be the Heaviside step function, 0 left of 0, and 1 right of 0. Since

$$(1 \ast \delta') \ast H = 1' \ast H = 0 \ast H = 0 \neq 1 = 1 \ast \delta = 1 \ast (\delta' \ast H)$$

associativity fails here.

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The spectral theory for normal compact operators on Hilbert spaces, and basic properties of Gelfand-Pettis integrals of vector-valued functions, have immediate application: uniqueness of invariant (Haar) measure on compact abelian groups $A$, and then proof that

$$L^2(A) = \text{completion of } \bigoplus_{\chi: A \to \mathbb{C}^\times} \mathbb{C} \cdot \chi$$

where $\chi$ runs over continuous characters of $A$, that is, continuous group homomorphisms $A \to \mathbb{C}^\times$. These characters arise as simultaneous eigenfunctions for the integral operators

$$T_\varphi : f \mapsto \int_A \varphi(y) f(x+y) \, dy \quad (\text{for } \varphi \in C_c^0(A) \text{ and } f \in L^2(A))$$

normalized to $\chi(0) = 1$, writing $A$ additively. This gives another approach to the $L^2$ theory of Fourier series on circles or products of circles, as well as harmonic analysis on the $p$-adic integers $\mathbb{Z}_p$, and more exotic items such as solenoids $A/\mathbb{Q}$, where $A$ is the adele group.

6. Approximate identities
One notion of approximate identity \( \{ \varphi_i \} \) on \( \mathbb{R}^n \) is a sequence of non-negative \( \varphi_i \in C_c^\infty(\mathbb{R}^n) \) whose supports shrink to \( \{ 0 \} \), in the sense that, given a neighborhood \( N \) of 0, there is \( i_0 \) such that for all \( i \geq i_0 \) the support of \( \varphi_i \) is inside \( N \). Further,

\[
\int_{\mathbb{R}^n} \varphi_i(g) \, dg = 1
\]

A less strict version replaces the shrinking of supports with the condition that, for every \( \varepsilon > 0 \) and \( \delta > 0 \), there is sufficiently large \( i_0 \) such that for every \( i \geq i_0 \)

\[
\int_{|x| \geq \delta} \varphi_i(g) \, dg < \varepsilon
\]

[6.1] Claim: For \( f \in L^2(\mathbb{R}^n) \) and for approximate identity \( \varphi_i \),

\[
\varphi_i \cdot f \longrightarrow f
\]

Proof: [... iou ...] ///