\documentclass{article}
\usepackage{amsmath, amssymb}
\title{07c. $C^\infty(\mathbb{T})$ is not normable}
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\begin{document}
From very general category-theory arguments, there is at most one projective-limit topology $C^\infty(\mathbb{T})$ by identifying countable limit of Banach spaces. We could take of a function on $T$ many natural function spaces, such as $C^3_a$. Maps from limits of Banach spaces to normed spaces factor through limitands $2$. Countable limits of Banach spaces $1$. Many natural function spaces, such as $C^\infty(a,b)$ and $C^\infty(\mathbb{T})$, are not Banach, nor even norm-able but still do have a metric topology and are complete: these are Fréchet spaces, appearing as countable (projective) limits of Banach spaces. It is reasonable to ask why these spaces are not Banach, and in fact not even normable, that is, their topologies cannot be given by a any norm, regardless of metric completeness.

In brief, in tangible terms, the root cause of this impossibility is that no estimates on the first $k$ derivatives of a function on $\mathbb{T}$ give an estimate on the $(k+1)^{th}$ derivative, for any $k$. This is discussed precisely below, and abstracted somewhat.

\section{1. Countable limits of Banach spaces}

We could take countable limit of Banach spaces as the definition of Fréchet space.

As earlier, $C^\infty(\mathbb{T})$ is a countable nested intersection, which is a countable (projective) limit:

$$C^\infty(\mathbb{T}) = \bigcap_{k \geq 0} C^k(\mathbb{T}) = \lim_{k \to \infty} C^k(\mathbb{T})$$

From very general category-theory arguments, there is at most one projective-limit topology on $C^\infty(\mathbb{T})$, up to unique isomorphism. Existence of the topology on $X$ satisfying the limit condition can be proven by identifying $X$ as the diagonal closed subspace of the topological product of the limitands $X_k$: letting $p_{k,k-1} : X_k \to X_{k-1}$ be the transition maps, $X = \{ \{x_k : x_k \in C^k[a,b]\} : p_{k,k-1}(x_k) = x_{k-1} \text{ for all } k \}$

The subspace topology on $X$ is the limit topology, seen as follows. The projection maps $p_k : \prod_j X_j \to X_k$ from the whole product to the factors $X_k$ are continuous, so their restrictions to the diagonally imbedded $X$ are continuous. Further, letting $i_k : X_k \to X_{k-1}$ be the transition map, on that diagonal copy of $X$ we have $i_k \circ p_k = p_{k-1}$ as required.

On the other hand, any family of maps $\varphi_k : Z \to X_k$ induces a map $\bar{\varphi} : Z \to \prod X_k$ such that $p_k \circ \bar{\varphi} = \varphi_k$, by the property of the product. Compatibility $i_k \circ \varphi_k = \varphi_{k-1}$ implies that the image of $\bar{\varphi}$ is inside the diagonal, that is, inside the copy of $X$. Thus, this construction does produce a limit.

A countable product of metric spaces $X_k$ with metrics $d_k$ has no canonical single metric, but is metrizable. One of many topologically equivalent metrics is the usual

$$d(\{x_k\}, \{y_k\}) = \sum_{k=0}^{\infty} 2^{-k} \frac{d_k(x_k - y_k)}{d_k(x_k - y_k) + 1}$$

When the metric spaces $X_k$ are complete, the product is complete. A closed subspace of a complete metrizable space is complete metrizable, so the diagonal $X$ is complete metric.

Even in general, the topologies on vector spaces $V$ are required to be translation invariant, meaning that for an open neighborhood $U$ of 0, for any $x \in V$, the set $x + U = \{x + u : u \in U\}$ is an open neighborhood of

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2. Maps from limits of Banach spaces to normed spaces factor through limitands

The assertion of the section title is only reliably true when the image of the limit in each limitand is dense. This hypothesis is unnecessary when the limitands are Hilbert spaces.

[2.1] Lemma: Given a continuous linear map $T$ from $C^\infty(\mathbb{T})$ to a normed space $Y$, there is an index $k$ such that when $C^\infty(\mathbb{T})$ is given the (weaker) $C^k$ topology, $T : C^\infty(\mathbb{T}) \to Y$ is still continuous.

[2.2] Corollary: Every continuous linear map $T$ from $C^\infty(\mathbb{T})$ to a Banach space $Y$ factors through some limitand $C^k(\mathbb{T})$. That is, there is $T_k : C^k(\mathbb{T}) \to Y$ such that $T = T_k \circ i_k$, where $i_k : C^\infty(\mathbb{T}) \to C^k(\mathbb{T})$ is the inclusion.

Proof: (of Corollary) After applying the lemma, since the target space of $T$ is complete, we can extend $T : C^\infty(\mathbb{T}) \to Y$ by continuity (in the $C^k$ topology) to the $C^k$-completion of $C^\infty$, which is $C^k$. ///

The lemma is a special case of the analogous lemma that has nothing to do with spaces of functions, but, rather, is true for more general reasons:

[2.3] Lemma: Let $X = \lim_k X_k$ be a limit of Banach spaces $X_k$, with projection maps $p_k : X \to X_k$. Suppose that $p_k(X)$ is dense in $X_k$. Then every continuous linear map $T : X \to Y$ to a normed space $Y$ factors through some limitand $X_k$. That is, there is $T_k : X_k \to Y$ such that $T = T_k \circ p_k$.

Proof: Given $\varepsilon > 0$, by the description above of the topology on the limit, there are $\delta > 0$ and index $k$ such that $T(U_{k,\delta})$ is inside the $\varepsilon$-ball at 0 in $Y$.

Then, given any other $\varepsilon' > 0$, we claim that $T$ maps

$$\frac{\varepsilon'}{\varepsilon} \cdot U_{k,\delta} = U_{k,\delta\varepsilon'/\varepsilon}$$

to the open $\varepsilon'$-ball in $Y$. Indeed,

$$|T(\frac{\varepsilon'}{\varepsilon} \cdot U_{k,\delta})|_Y = \frac{\varepsilon'}{\varepsilon} \cdot |T(U_{k,\delta})|_Y < \frac{\varepsilon'}{\varepsilon} \cdot \varepsilon = \varepsilon'$$

as claimed. Thus, $T : X \to Y$ is continuous when $X$ is given the $X_k$ topology, for the index $k$ that makes this work. Thus, $T$ extends by continuity to the $|\cdot|_{X_k}$-completion of $X$. By the density assumption, this is $X_i$. ///

[2.4] Remark: Finite Fourier series, which are in $C^\infty(\mathbb{T})$, are dense in every $C^k(\mathbb{T})$, so $C^\infty(\mathbb{T})$ is dense in every $C^k(\mathbb{T})$.

[2.5] Remark: In the case that $Y = \mathbb{C}$, the density assumption is unnecessary, since Hahn-Banach gives an extension. But for general Banach $Y$, without the density assumption, we can only conclude that $T$ factors through the $|\cdot|_{X_i}$-completion of $X$, since not all closed subspaces of Banach spaces are complemented.

[1] For Hilbert and Banach spaces, this translation-invariance is clear, since the topology is metric, and comes from a norm.
3. $C^\infty(\mathbb{T})$ is not normable

If $C^\infty(\mathbb{T})$ were normable, then the identity map $j : C^\infty(\mathbb{T}) \to C^\infty(\mathbb{T})$ would be continuous when the source is given the $C^k$ topology. In particular, for every $\varepsilon > 0$, there would be a sufficiently small $\| \cdot \|_{X_k}$-ball $B$ whose image in $C^\infty(\mathbb{T})$ under the inclusion is inside the $\varepsilon$-ball in the $C^{k+1}(\mathbb{T})$ topology on $C^\infty(\mathbb{T})$. Specifically, for $\varepsilon = 1$, there should be a sufficiently small $\delta > 0$ such that the $\delta$-ball in the $C^k$ topology is inside the unit ball in the $C^{k+1}$ topology.

However, it is easy-enough to construct $C^\infty$ functions whose $C^k$ norms are arbitrarily small, but whose $C^{k+1}$ norm is 1, for example, $e^{iN x}/N^{k+1}$. Thus, we achieve a contradiction.