Introduction to Levi-Sobolev spaces

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[This document is http://www.math.umn.edu/~garrett/m/realfur/notes_2018-19/08a_intro_Sobolev.pdf]

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1. Prototypical Sobolev imbedding theorem

The simplest case of a Levi-Sobolev imbedding theorem asserts that the +1-index Levi-Sobolev Hilbert space $H^1[a,b]$ described below is inside $C^0[a,b]$. This is a corollary of a Levi-Sobolev inequality asserting that the $C^0[a,b]$ norm is dominated by the $H^1[a,b]$ norm. All that is used is the fundamental theorem of calculus and the Cauchy-Schwarz-Bunyakowsky inequality. The point is that there is a large Hilbert space $H^1[a,b]$ inside the Banach space $C^0[a,b]$.

We will do much more with this idea subsequently.

We can think of $L^2[a,b]$ as

$$L^2[a,b] = \text{completion of } C^0[a,b] \text{ with respect to } |f|_{L^2} = \left( \int_a^b |f(t)|^2 \, dt \right)^{1/2}$$

The +1-index Levi-Sobolev space\([1]\) $H^1[a,b]$ is

$$H^1[a,b] = \text{completion of } C^1[a,b] \text{ with respect to } |f|_{H^1} = \left( |f|_{L^2[a,b]}^2 + |f'|_{L^2[a,b]}^2 \right)^{1/2}$$

\[1.1\] Theorem: (Levi-Sobolev inequality) On $C^1[a,b]$, the $H^1[a,b]$-norm dominates the $C^0[a,b]$-norm. That is, there is a constant $C$ depending only on $a,b$ such that $|f|_{C^0[a,b]} \leq C \cdot |f|_{H^1[a,b]}$ for every $f \in C^1[a,b]$.

Proof: For $a \leq x \leq y \leq b$, for $f \in C^1[a,b]$, the fundamental theorem of calculus gives

$$|f(y) - f(x)| = \left| \int_x^y f'(t) \, dt \right| \leq \int_x^y |f'(t)| \, dt \leq \left( \int_x^y |f'(t)|^2 \, dt \right)^{1/2} \cdot \left( \int_x^y 1 \, dt \right)^{1/2}$$

$$\leq |f'_{L^2} \cdot |x - y|^{1/2} \leq |f'_{L^2} \cdot |a - b|^{1/2}$$

Using the continuity of $f \in C^1[a,b]$, let $y \in [a,b]$ be such that $|f(y)| = \min_{x} |f(x)|$. Using the previous inequality,

$$|f(x)| \leq |f(y)| + |f(x) - f(y)| \leq \frac{1}{|a - b|} \int_a^b |f(t)| \, dt + |f(x) - f(y)| \leq \frac{1}{|a - b|} \int_a^b |f| \cdot 1 \, dt + |f'_{L^2} \cdot |a - b|^{1/2}$$

$$\leq \frac{|f|_{L^2}^{1/2} \cdot |a - b|^{1/2}}{|a - b|} + |f'_{L^2} \cdot |a - b|^{1/2} = \frac{|f|_{L^2}^{1/2}}{|a - b|^{1/2}} + |f'_{L^2} \cdot |a - b|^{1/2} \leq \left( |f|_{L^2} + |f'|_{L^2} \right) \cdot \left( |a - b|^{-1/2} + |a - b|^{1/2} \right)$$

$$\leq 2(|f|_{L^2} + |f'|_{L^2})^{1/2} \cdot \left( |a - b|^{-1/2} + |a - b|^{1/2} \right) = |f|_{H^1} \cdot 2 \left( |a - b|^{-1/2} + |a - b|^{1/2} \right)$$

\[1\] ... also denoted $W^{1,2}[a,b]$, where the superscript 2 refers to $L^2$, rather than $L^p$. Beppo Levi noted the importance of taking Hilbert space completion with respect to this norm in 1906, giving a correct formulation of Dirichlet’s principle. Sobolev’s systematic development of these ideas was in the mid-1930’s.
Thus, on $C^1[a,b]$ the $H^1$ norm dominates the $C^\infty$-norm.

\[\text{[1.2] Corollary: (Levi-Sobolev imbedding) } H^1[a,b] \subset C^\infty[a,b].\]

\text{Proof:} Since $H^1[a,b]$ is the $H^1$-norm completion of $C^1[a,b]$, every $f \in H^1[a,b]$ is an $H^1$-limit of functions $f_n \in C^1[a,b]$. That is, $|f - f_n|_{H^1[a,b]} \to 0$. Since the $H^1$-norm dominates the $C^\infty$-norm, $|f - f_n|_{C^\infty[a,b]} \to 0$. A $C^\infty$ limit of continuous functions is continuous, so $f$ is continuous.

In fact, we have a stronger conclusion than continuity, namely, a \textit{Lipschitz condition} with exponent $\frac{1}{2}$:

\[\text{[1.3] Corollary: (of proof of theorem) } |f(x) - f(y)| \leq |f'|_{L^2} \cdot |x - y|^{\frac{1}{2}} \text{ for } f \in H^1[a,b].\]

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## 2. Sobolev theorems on $\mathbb{T}^n$

For $0 \leq k \in \mathbb{Z}$ and $f \in C^\infty(\mathbb{T}^n)$, the $k^{th}$ Sobolev norm can be defined in terms of $L^2$ norms of all its derivatives up through order $k$:

$$|f|_{H^k}^2 = \sum_{|\alpha| \leq k} |f^{(\alpha)}|_{L^2}^2$$

where as usual $\alpha$ is summed over multi-indices $\alpha = (\alpha_1, \ldots, \alpha_n)$ of non-negative integers $\alpha_i$, with $|\alpha| = \alpha_1 + \ldots + \alpha_n$, and

$$f^{(\alpha)} = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}} f$$

Then one way to define the $k^{th}$ Sobolev space is

$$H^k(\mathbb{T}^n) = \text{completion of } C^\infty(\mathbb{T}^n) \text{ with respect to } |\cdot|_{H^k}$$

In this context, $H^{-k}(\mathbb{T}^n)$ for $-k < 0$ is defined to be the dual of $H^k(\mathbb{T}^n)$, with $H^0(\mathbb{T}^n) = L^2(\mathbb{T}^n)$ identified with itself via Riesz-Fréchet (and pointwise conjugation, so that Riesz-Fréchet gives a $\mathbb{C}$-linear isomorphism rather than $\mathbb{C}$-conjugate-linear). From the inclusion $H^{k+1} \to H^k$ for $0 \leq k \in \mathbb{Z}$ dualizing gives a dual/adjoint map $H^{-k} \to H^{-k-1}$. Let

$$H^\infty(\mathbb{T}^n) = \bigcap_{k=0}^{\infty} H^k(\mathbb{T}^n) = \lim_k H^k(\mathbb{T}^n)$$

and

$$H^{-\infty}(\mathbb{T}^n) = \bigcup_{k=0}^{\infty} H^{-k}(\mathbb{T}^n) = \text{colim}_k H^{-k}(\mathbb{T}^n)$$

The picture is

$$H^\infty(\mathbb{T}^n) \to H^1(\mathbb{T}^n) \to H^0(\mathbb{T}^n) \to H^{-1}(\mathbb{T}^n) \to \cdots \to H^{-\infty}(\mathbb{T}^n)$$

\[\text{[2.1] Claim: All arrows are continuous injections with dense images.}\]

\text{Proof: } [...] iou [...] \quad \///

A \textit{spectral characterization} of Sobolev norms is often useful, and directly defines $H^s(\mathbb{T}^n)$ for all $s \in \mathbb{R}$: for $f \in C^\infty(\mathbb{T}^n)$, with Fourier coefficients $\hat{f}(\xi)$,

$$|f|_{H^s}^2 = \sum_{\xi \in \mathbb{Z}^n} |\hat{f}(\xi)|^2 \cdot (1 + |\xi|^2)^s$$

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and $H^s(\mathbb{T}^n)$ is the completion of $C^\infty(\mathbb{T}^n)$ with this norm.

**[2.2] Claim:** The spectral characterization gives the same topology on $H^k$ as the characterization in terms of $L^2$ norms of derivatives, for $0 \leq k \in \mathbb{Z}$.

**Proof:** [...] //

Sometimes it is convenient to give the derivative characterization slightly differently, as

$$|f|_{H^k}^2 = \langle (1 - \Delta)^k f, f \rangle_{L^2}$$

**[2.3] Claim:** The latter characterization gives the same topology on $H^k$ as do the two previous characterizations, for $0 \leq k \in \mathbb{Z}$.

**Proof:** [...] //

**[2.4] Theorem:** (Sobolev imbedding theorem) $H^s(\mathbb{T}^n) \subset C^k(\mathbb{T}^n)$ for $s > \frac{n}{2}$.

**Proof:** [...] //

**[2.5] Corollary:** $H^\infty(\mathbb{T}^n) \subset C^\infty(\mathbb{T}^n)$, and $H^{-\infty}(\mathbb{T}^n) = C^\infty(\mathbb{T}^n)^*$.

**Proof:** [...] //

**[2.6] Theorem:** The duality pairing $H^s \times H^{-s} \to \mathbb{C}$ can also be given by an extension of Plancherel, namely, for $\psi_\xi(x) = e^{2\pi i \xi \cdot x}$,

$$\langle \sum_\xi a_\xi \psi_\xi, \sum_\xi b_\xi \psi_\xi \rangle_{H^s \times H^{-s}} = \sum_\xi a_\xi \cdot \overline{b_\xi}$$

**Proof:** [...] //

That is, distributions on $\mathbb{T}^n$ admit Fourier expansions with coefficients of moderate growth, and evaluation of distributions on smooth functions can be done by a natural extension of Plancherel.

### 3. Sobolev theorems on $\mathbb{R}^n$

The general shape of the discussion on $\mathbb{R}^n$ is similar to that on $\mathbb{T}^n$, with some unsurprising complications due to the non-compactness of $\mathbb{R}$. In particular, Fourier series are replaced by Fourier transforms and inversion.

For $0 \leq k \in \mathbb{Z}$ and $f \in C^\infty_c(\mathbb{R}^n)$, the $k^\text{th}$ Sobolev norm can be defined in terms of $L^2$ norms of all its derivatives up through order $k$:

$$|f|_{H^k}^2 = \sum_{|\alpha| \leq k} |f^{(\alpha)}|_{L^2}$$

One way to define the $k^\text{th}$ Sobolev space is

$$H^k(\mathbb{R}^n) = \text{completion of } C^\infty_c(\mathbb{R}^n) \text{ with respect to } | \cdot |_{H^k}$$

In this context, $H^{-k}(\mathbb{R}^n)$ for $-k < 0$ is defined to be the dual of $H^k(\mathbb{R}^n)$, with $H^0(\mathbb{R}^n) = L^2(\mathbb{R}^n)$ identified with itself via Riesz-Fréchet (and pointwise conjugation, so that Riesz-Fréchet gives a $\mathbb{C}$-linear isomorphism
rather than \( C\)-conjugate-linear). From the inclusion \( H^{k+1} \to H^k \) for \( 0 \leq k \in \mathbb{Z} \) dualizing gives a dual/adjoint map \( H^{-k} \to H^{-k-1} \). Let
\[
H^\infty(\mathbb{R}^n) = \bigcap_{k=0}^{\infty} H^k(\mathbb{R}^n) = \lim_{k} H^k(\mathbb{R}^n)
\]
and
\[
H^{-\infty}(\mathbb{R}^n) = \bigcup_{k=0}^{\infty} H^{-k}(\mathbb{R}^n) = \text{colim}_k H^{-k}(\mathbb{R}^n)
\]
The picture is the same as for \( T^n \):

\[
H^\infty(\mathbb{R}^n) \to \ldots \to H^1(\mathbb{R}^n) \to H^0(\mathbb{R}^n) \to H^{-1}(\mathbb{R}^n) \to \ldots \to H^{-\infty}(\mathbb{R}^n)
\]

[3.1] Claim: All arrows are continuous injections with dense images.

Proof: [...] iou ...} ///

A spectral characterization of Sobolev norms is often useful, and directly defines \( H^s(\mathbb{R}^n) \) for all \( s \in \mathbb{R} \): for \( f \in C^\infty_c(\mathbb{R}^n) \), with Fourier transform \( \hat{f}(\xi) \),
\[
|f|_{H^s}^2 = \int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 \cdot (1 + |\xi|^2)^s \, d\xi
\]
and \( H^s(\mathbb{R}^n) \) is the completion of \( C^\infty_c(\mathbb{R}^n) \) with this norm.

[3.2] Claim: The spectral characterization gives the same topology on \( H^k \) as the characterization in terms of \( L^2 \) norms of derivatives, for \( 0 \leq k \in \mathbb{Z} \).

Proof: [...] iou ...} ///

[3.3] Corollary: Distributions \( u \) in \( H^{-\infty}(\mathbb{R}^n) \) have Fourier transforms that are in weighted \( L^2 \) spaces, with pointwise values almost everywhere.

Proof: [...] iou ...} ///

Sometimes it is convenient to give the derivative characterization slightly differently, as
\[
|f|_{H^k}^2 = \langle (1 - \Delta)^k f, f \rangle_{L^2}
\]

[3.4] Claim: The latter characterization gives the same topology on \( H^k \) as do the two previous characterizations, for \( 0 \leq k \in \mathbb{Z} \).

Proof: [...] iou ...} ///

[3.5] Theorem: (Sobolev imbedding theorem) \( H^s(\mathbb{R}^n) \subset C^k(\mathbb{R}^n) \) for \( s > \frac{k}{2} \).

Proof: [...] iou ...} ///

Since \( \mathbb{R}^n \) is non-compact, the conclusion of the following is weaker than for \( T^n \), since \( C^\infty(\mathbb{R}^n) \) is not equal to \( \mathcal{S}(\mathbb{R}^n) \) or \( D(\mathbb{R}^n) \):

[3.6] Corollary: \( H^\infty(\mathbb{R}^n) \subset C^\infty(\mathbb{R}^n) \) and \( H^{-\infty}(\mathbb{R}^n) \supset C^\infty(\mathbb{R}^n)^* \).
Proof: [...] iou ...] ///

[3.7] Corollary: If we know that $\mathcal{E}(\mathbb{R}^n)^* = C^\infty(\mathbb{R}^n)^*$ is exactly compactly-supported distributions, then we can conclude that $H^{-\infty}(\mathbb{R}^n)$ contains compactly-supported distributions. /////

[3.8] Theorem: The duality pairing $H^s \times H^{-s} \to \mathbb{C}$ can also be given by an extension of Plancherel, namely, for $\psi_\xi(x) = e^{2\pi i \xi \cdot x}$,

$$
\langle f, F \rangle_{H^s \times H^{-s}} = \int_{\mathbb{R}^n} \hat{f}(\xi) \cdot \overline{\hat{F}(\xi)} \, d\xi
$$

Proof: [...] iou [...] ///

That is, evaluation of distributions in $H^{-\infty}(\mathbb{R}^n)$ on smooth functions in $H^\infty(\mathbb{R}^n)$ can be done by a natural extension of Plancherel.