1. **Euclidean spaces**

Let $\mathbb{R}^n$ be the usual Euclidean $n$-space, that is, ordered $n$-tuples $x = (x_1, \ldots, x_n)$ of real numbers. In addition to vector addition (termwise) and scalar multiplication, we have the usual distance function on $\mathbb{R}^n$, in coordinates $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$, defined by

$$d(x, y) = \sqrt{(x_1 - y_1)^2 + \ldots + (x_n - y_n)^2}$$

Of course there is visible symmetry $d(x, y) = d(y, x)$, and positivity: $d(x, y) = 0$ only for $x = y$. The triangle inequality

$$d(x, z) \leq d(x, y) + d(y, z)$$

is not trivial to prove. In the one-dimensional case, the triangle inequality is an inequality on absolute values, and can be proven case-by-case. In $\mathbb{R}^n$, it is best to use the following set-up. The usual *inner product* (or *dot-product*) on $\mathbb{R}^n$ is

$$x \cdot y = \langle x, y \rangle = \langle (x_1, \ldots, x_n), (y_1, \ldots, y_n) \rangle = x_1y_1 + \ldots + x_ny_n$$

and the length or norm of $x$ is $|x| = |x, x|^2$. *Context* distinguishes the norm $|x|$ of $x \in \mathbb{R}^n$ from the usual absolute value $|c|$ on real or complex numbers $c$. Distance from $x$ to $y$ is

$$d(x, y) = |x - y|$$

The inner product $\langle x, y \rangle$ is linear in both arguments: in the first argument

$$\langle x + x', y \rangle = \langle x, y \rangle + \langle x', y \rangle \quad \langle cx, y \rangle = c \cdot \langle x, y \rangle \quad \text{for } x, x', y \in \mathbb{R}^n \text{ and scalar } c$$

and similarly for the second argument. The triangle inequality will be a corollary of the following universally-useful inequality:
[1.1] **Claim:** *(Cauchy-Schwarz-Bunyakowsky inequality)* For \( x, y \in \mathbb{R}^n \),

\[
|\langle x, y \rangle| \leq |x| \cdot |y|
\]

Assuming that neither \( x \) nor \( y \) is 0, strict inequality holds unless \( x \) and \( y \) are scalar multiples of each other.

**Proof:** If \( |y| = 0 \), the assertions are trivially true. Thus, take \( y \neq 0 \). With real \( t \), consider the quadratic polynomial function

\[
f(t) = |x - ty|^2 = |x|^2 - 2t\langle x, y \rangle + t^2|y|^2
\]

Certainly \( f(t) \geq 0 \) for all \( t \in \mathbb{R} \), since \( |x - ty| \geq 0 \). Its minimum occurs where \( f'(t) = 0 \), namely, where

\[
-2\langle x, y \rangle + 2t|y|^2 = 0.
\]

This is where \( t = \langle x, y \rangle / |y|^2 \). Thus,

\[
0 \leq (\text{minimum}) \leq f(\langle x, y \rangle / |y|^2) = |x|^2 - 2\frac{\langle x, y \rangle}{|y|^2}\langle x, y \rangle + \left( \frac{\langle x, y \rangle}{|y|^2} \right)^2 \cdot |y|^2 = |x|^2 - \left( \frac{\langle x, y \rangle}{|y|^2} \right)^2 \cdot |y|^2
\]

Multiplying out by \( |y|^2 \),

\[
0 \leq |x|^2 \cdot |y|^2 - \langle x, y \rangle^2
\]

which gives the inequality. Further, for the inequality to be an *equality*, it must be that \( |x - ty| = 0 \), so \( x \) is a multiple of \( y \).

///

[1.2] **Remark:** We did not use properties of \( \mathbb{R}^n \), only of the inner product!

[1.3] **Corollary:** *(Triangle inequality)* For \( x, y, z \in \mathbb{R}^n \),

\[
|x + y| \leq |x| + |y|
\]

Therefore,

\[
d(x, z) = |x - z| = |(x - y) - (z - y)| \leq |x - y| + |z - y| = d(x, y) + d(y, z)
\]

**Proof:** With the Cauchy-Schwarz-Bunyakowsky inequality in hand, this is a direct computation:

\[
|x + y|^2 = \langle x + y, x + y \rangle = \langle x, x \rangle + \langle x, y \rangle + \langle y, x \rangle + \langle y, y \rangle = |x|^2 + 2\langle x, y \rangle + |y|^2 \leq |x|^2 + 2|\langle x, y \rangle| + |y|^2
\]

\[
\leq |x|^2 + 2|x| \cdot |y| + |y|^2 = (|x| + |y|)^2
\]

Taking positive square roots gives the result. ///

As usual, the *open ball* of radius \( r > 0 \) centered at a point \( y \) is

\[
\text{open ball} = \{ x \in \mathbb{R}^n : d(x, y) < r \}
\]

The *closed ball* of radius \( r > 0 \) centered at a point \( y \) is

\[
\text{closed ball} = \{ x \in \mathbb{R}^n : d(x, y) \leq r \}
\]

Obviously in many regards the two are barely different from each other. However, the fact that the *closed* ball includes its *boundary* (in both an intuitive an technical sense as below), namely, the sphere

\[
S^{n-1} = \{ x \in \mathbb{R}^n : d(x, y) = r \}
\]

while the *open* ball does *not*. A different distinction is what we’ll exploit most directly:
[1.4] **Corollary:** For any point $x$ in an open ball $B$ in $\mathbb{R}^n$, for sufficiently small radius $\varepsilon > 0$ the open ball of radius $\varepsilon$ centered at $x$ is contained in $B$.

**Proof:** This is essentially the triangle inequality. Let $B$ be the open ball of radius $r$ centered at $y$. Then $x \in B$ if and only if $|x - y| < r$. Thus, we can take $\varepsilon > 0$ such that $|x - y| + \varepsilon < r$. For $|z - x| < \varepsilon$, by the triangle inequality

$$|z - y| \leq |z - x| + |x - y| < \varepsilon + |x - y| < r$$

That is, the open ball of radius $\varepsilon$ at $x$ is inside $B$. ///

An open set in $\mathbb{R}^n$ is any set with the property observed in the latter corollary, namely a set $U$ in $\mathbb{R}^n$ is open if for every $x$ in $U$ there is an open ball centered at $x$ contained in $U$.

This definition allows us to rewrite the epsilon-delta definition of continuity in a useful form:

[1.5] **Claim:** A function $f : \mathbb{R}^m \to \mathbb{R}^n$ is continuous if and only if the inverse image

$$f^{-1}(U) = \{x \in \mathbb{R}^m : f(x) \in U\}$$

of every open set $U$ in $\mathbb{R}^n$ is open in $\mathbb{R}^m$. (We prove this below for general metric spaces.) ///

Some properties of open sets in $\mathbb{R}^n$ that will be abstracted:

[1.6] **Claim:** The union of an arbitrary set of open subsets of $\mathbb{R}^n$ is open. The intersection of a finite set of open subsets of $\mathbb{R}^n$ is open.

**Proof:** A point $x \in \mathbb{R}^n$ is in the union $U$ of an arbitrary set $\{U_\alpha : \alpha \in A\}$ of open subsets of $\mathbb{R}^n$ exactly when there is some $U_\alpha$ so that $x \in U_\alpha$. Then a small-enough open ball $B$ centered at $x$ is inside $U_\alpha$, so $B \subset U_\alpha \subset U$.

For $x$ in the intersection $I = U_1 \cap \ldots \cap U_m$ of a finite number of opens, let $\varepsilon_j > 0$ such that the open $\varepsilon_j$-ball at $x$ is contained in $U_j$. Let $\varepsilon$ be the minimum of the $\varepsilon_j$. The minimum of a finite set of (strictly) positive real numbers is still (strictly) positive, so $\varepsilon > 0$, and the $\varepsilon$-ball at $x$ is contained inside every $\varepsilon_j$-ball at $x$, so is contained in the intersection. ///

One of many equivalent ways to say that a set $E$ in $\mathbb{R}^n$ is bounded is that it is contained in some (sufficiently large) ball. [1] At various technical points in advanced calculus, we find ourselves caring about closed and bounded sets, and perhaps proving the Heine-Borel property or Bolzano-Weierstraß property.[2]

[1.7] **Theorem:** A set $E$ in $\mathbb{R}^n$ is closed and bounded if and only if every sequence of points in $E$ has a convergent subsequence. ///

---

### 2. Metric spaces

By design, the previous discussion of Euclidean spaces made minimal use of particular features of Euclidean space. This allows abstraction in a manner using intuition about finite-dimensional Euclidean spaces to suggest things about less familiar spaces. The process of abstraction has several stopping places, and this section looks at one of the first.

[1] A few moments’ thought show that it does not matter where the ball is centered, nor whether the ball is closed or open.

[2] This property is not at all trivial to prove, especially from an elementary viewpoint.
We can abstract the *distance function* on \( \mathbb{R}^n \) usefully, as follows. For a set \( X \) be a set, a non-negative-real-valued function

\[
d : X \times X \to \mathbb{R}
\]
is a *distance function* if it satisfies the conditions

\[
\begin{align*}
    d(x, y) &\geq 0 \quad \text{(with equality only for} \ x = y) \quad \text{(positivity)} \\
    d(x, y) &= d(y, x) \quad \text{(symmetry)} \\
    d(x, z) &\leq d(x, y) + d(y, z) \quad \text{(triangle inequality)}
\end{align*}
\]

for all points \( x, y, z \in X \). Such a distance function is also called a *metric*. The set \( X \) with the metric \( d \) is a *metric space*.

In analogy with the situation for \( \mathbb{R} \) and \( \mathbb{R}^n \), a sequence \( \{x_n\} \) in a metric space \( X \) is *convergent* to \( x \in X \) when, for every \( \varepsilon > 0 \), there is \( n_0 \) such that, for all \( n \geq n_0 \), \( |x_n - x| < \varepsilon \). Likewise, a sequence \( \{x_n\} \) in \( X \) is a *Cauchy* sequence when, for all \( \varepsilon > 0 \), there is \( n_0 \) such that for all \( m, n \geq n_0 \), \( |x_m - x_n| < \varepsilon \). A metric space is *complete* if every Cauchy sequence is convergent.

The following standard lemma makes a bit of intuition explicit:

**[2.1] Lemma:** Let \( \{x_i\} \) be a Cauchy sequence in a metric space \( X \), \( d \) converging to \( x \) in \( X \). Given \( \varepsilon > 0 \), let \( N \) be sufficiently large such that \( d(x_i, x_j) < \varepsilon \) for \( i, j \geq N \). Then \( d(x_i, x) \leq \varepsilon \) for \( i \geq N \).

**Proof:** Let \( \delta > 0 \) and take \( j \geq N \) also large enough such that \( d(x_j, x) < \delta \). Then for \( i \geq N \) by the triangle inequality

\[
d(x_i, x) \leq d(x_i, x_j) + d(x_j, x) < \varepsilon + \delta
\]

Since this holds for every \( \delta > 0 \) we have the result. ///

**[2.2] Example:** Variants of the usual Euclidean metric on \( \mathbb{R}^n \) also make sense:

\[
d_1(x, y) = |x_1 - y_1| + \ldots + |x_n - y_n| \quad \quad d_\infty(x, y) = \max_i |x_i - y_i|
\]

In fact, the triangle inequality for these metrics are easy to prove, needing just the triangle inequality for the absolute value on \( \mathbb{R} \). Later, we will see\(^3\) that

\[
d_p(x, y) = \left( |x_1 - y_1|^p + \ldots + |x_n - y_n|^p \right)^{1/p} \quad \text{(for} \ 1 \leq p < \infty)
\]

also gives a metric.

**[2.3] Example:** A *discrete set* or *discrete metric space* \( X \) is one in which (roughly) no two distinct points are close to each other. That is, for each \( x \in X \) there should be a bound \( \delta_x > 0 \) such that \( d(x, y) \geq \delta_x \) for all \( y \neq x \) in \( X \). For example, the set \( \mathbb{Z} \) of integers, with the natural distance

\[
d(x, y) = |x - y| \quad \quad \text{(with usual absolute value)}
\]

has the property that \( |x - y| \geq 1 \) for distinct integers. Every discrete metric space is complete.

**[2.4] Example:** Any set \( X \) can be made into a *discrete* metric space by defining

\[
d(x, y) = \begin{cases} 
    1 & \text{(for} \ x \neq y) \\
    0 & \text{(for} \ x = y)
\end{cases}
\]

\(^3\) The triangle inequality for such metrics is an instance of the Hölder inequality.
This is obviously positive and symmetric, and satisfies the triangle inequality condition for silly reasons. Little is learned from this example except that it is possible to do such things.

[2.5] Example: The collection $C^0[a, b]$ of continuous functions on an interval $[a, b]$ on the real line can be given the metric

$$d(f, g) = \sup_{x \in [a, b]} |f(x) - g(x)|$$

Positivity and symmetry are easy, and the triangle inequality is not hard, either. This metric space is complete, because a Cauchy sequence is a uniformly pointwise convergent sequence of continuous functions.

[2.6] Example: The collection $C^0(\mathbb{R})$ of continuous functions on the whole real line does not have an obvious candidate for a metric, since the $\sup$ metric of the previous example may give infinite values. Yet there is the metric

$$d(f, g) = \sum_{n=1}^{\infty} 2^{-n} \frac{\sup_{|x| \leq n} |f(x) - g(x)|}{1 + \sup_{|x| \leq n} |f(x) - g(x)|}$$

This metric space is complete, for similar reasons as $C^0[a, b]$.

[2.7] Example: A sort of infinite-dimensional analogue of the standard metric on $\mathbb{R}^n$ is the space $\ell^2$, the collection of all sequences $\alpha = (\alpha_1, \alpha_2, \ldots)$ of complex numbers such that $\sum_{n=1}^{\infty} |\alpha_n|^2 < +\infty$. The metric is

$$d(\alpha, \beta) = \sqrt{\sum_{n=1}^{\infty} |\alpha_n - \beta_n|^2}$$

In fact, $\ell^2$ is a vector space, being closed under addition and under scalar multiplication, with inner product

$$\langle \alpha, \beta \rangle = \sum_{n=1}^{\infty} \alpha_n \cdot \overline{\beta_n}$$

Cauchy-Schwarz-Bunyakowsky shows that the latter sum converges absolutely. The associated norm is $|\alpha| = \langle \alpha, \alpha \rangle^{1/2}$, and $d(\alpha, \beta) = |\alpha - \beta|$. The Cauchy-Schwarz-Bunyakowsky holds for $\ell^2$, by the same proof as given earlier, and proves the triangle inequality. This metric space is complete, as we show a little later.

[2.8] Example: For $1 \leq p < \infty$, the sequence space $\ell^p$ is

$$\ell^p = \{ x = (x_1, x_2, \ldots) : \sum_{i=1}^{\infty} |x_i|^p < \infty \}$$

with metric

$$d_p(x, y) = \left( \sum_{i=1}^{\infty} |x_i - y_i|^p \right)^{1/p}$$

Proof of the triangle inequality needs Hölder’s inequality, and we will take care of this later. These metric spaces are complete, as we see later. Unlike the case of varying metrics on $\mathbb{R}^n$, the underlying sets $\ell^p$ are not the same, and the topologies are all different. For example, $\ell^2$ is strictly larger than $\ell^1$.

[2.9] Example: Even before having a modern notion of measure and integral, an analogue of $\ell^2$ can be formulated: on $C^0[a, b]$, form an inner product

$$\langle f, g \rangle = \int_a^b f(x) \overline{g(x)} \, dx$$

[4] Throughout discussion of these examples, it doesn’t matter much whether we think of real-valued functions or complex-valued functions.

[5] Real-valued or complex-valued, for example.
It is easy to check that this does give a hermitian inner product. The $L^2$ norm is $|f|_{L^2} = \langle f, f \rangle^{\frac{1}{2}}$, and the distance function is $d(f, g) = |f - g|$. The basic properties of a metric are immediate, except that the triangle inequality needs the integral form of the Cauchy-Schwarz-Bunyakowsky inequality, whose proof is the same as that given earlier. This metric space is not complete, because there are sequences of continuous functions that are Cauchy in this $L^2$ metric (but not in the $C^0[a,b]$ metric) and do not converge to a continuous function. For example, we can make piecewise-linear continuous functions approaching the discontinuous function that is 0 on $[a, a + b/2)$ and 1 on $[a + b/2, b]$, by

$$f_n(x) = \begin{cases} 0 & (\text{for } a \leq x \leq \frac{a + b}{2} - \frac{1}{n}) \\ \frac{n}{2} \cdot (x - (\frac{a + b}{2} - \frac{1}{n})) & (\text{for } \frac{a + b}{2} - \frac{1}{n} \leq x \leq \frac{a + b}{2} + \frac{1}{n}) \\ 1 & (\text{for } \frac{a + b}{2} + \frac{1}{n} \leq x \leq b) \end{cases}$$

(Draw a picture.) The pointwise limit is 0 to the left of the midpoint, and 1 to the right. Despite the fact that the pointwise limit does not exist at the midpoint,

$$d_2(f_i, f_j)^2 \leq \int_{\frac{a + b}{2} - \frac{1}{n}}^{\frac{a + b}{2} + \frac{1}{n}} 1 \, dx \leq \frac{2}{n} \quad (\text{for } i, j \geq n)$$

which goes to 0 as $n \to \infty$. That is, $\{f_n\}$ is Cauchy in the $L^2$ metric, but does not converge to a continuous function.

### 3. Topologies of metric spaces

The notion of metric space allows a useful generalization of the notion of continuous function via the obvious analogue of the epsilon-delta definition:

A function or map $f : X \to Y$ from one metric space $(X, d_X)$ to another metric space $(Y, d_Y)$ is continuous at a point $x_0 \in X$ when, for every $\varepsilon > 0$ there is $\delta > 0$ such that

$$d_X(x, x_0) < \delta \quad \implies \quad d_Y(f(x), f(x_0)) < \varepsilon$$

In a metric space $(X, d)$, the open ball of radius $r > 0$ centered at a point $y$ is

$$\{x \in X : d(x, y) < r\}$$

The closed ball of radius $r > 0$ centered at a point $y$ is

$$\{x \in X : d(x, y) \leq r\}$$

As in $\mathbb{R}^n$, in many regards the two are barely different from each other. However, the closed ball includes the sphere

$$\{x \in X : d(x, y) = r\}$$

while the open ball does not. A different distinction is what we’ll exploit most directly:

[3.1] **Claim:** For any point $x$ in an open ball $B$ in $X$, for sufficiently small radius $\varepsilon > 0$ the open ball of radius $\varepsilon$ centered at $x$ is contained in $B$. (As for $\mathbb{R}^n$, this follows immediately by use of the triangle inequality. //)

An open set in $X$ is any set with the property observed in this proposition. That is, a set $U$ in $X$ is open if for every $x$ in $U$ there is an open ball centered at $x$ contained in $U$. 


This definition allows us to rewrite the epsilon-delta definition of continuity in a form that will apply in more general topological spaces:

**[3.2] Claim:** A function \( f : X \to Y \) from one metric space to another is continuous in the \( \varepsilon\)-\( \delta \) sense if and only if the inverse image

\[
 f^{-1}(U) = \{ x \in \mathbb{R}^m : f(x) \in U \}
\]

of every open set \( U \) in \( Y \) is open in \( X \).

**Proof:** On one hand, suppose \( f \) is continuous in the \( \varepsilon\)-\( \delta \) sense. For \( U \) open in \( Y \) and \( x \in f^{-1}(U) \), with \( f(x) = y \), let \( \varepsilon > 0 \) be small enough so that the \( \varepsilon \)-ball at \( y \) is inside \( U \). Take \( \delta > 0 \) small enough so that, by the \( \varepsilon\)-\( \delta \) definition of continuity, the \( \delta \)-ball \( B \) at \( x \) has image \( f(B) \) inside the \( \varepsilon \)-ball at \( y \). Then \( x \in B \subseteq f^{-1}(U) \). This holds for every \( x \in f^{-1}(U) \), so \( f^{-1}(U) \) is open.

On the other hand, suppose \( f^{-1}(U) \) is open for every open \( U \subset Y \). Given \( x \in X \) and \( \varepsilon > 0 \), let \( U \) be the \( \varepsilon \)-ball at \( f(x) \). Since \( f^{-1}(U) \) is open, there is an open ball \( B \) at \( x \) contained in \( f^{-1}(U) \). Let \( \delta > 0 \) be the radius of \( B \).

A set \( E \) in a metric space \( X \) is closed if and only its complement

\[
 E^c = X - E = \{ x \in X : x \notin E \}
\]

is open.

A set \( E \) in a metric space \( X \) is bounded when it is contained in some (sufficiently large) ball. This makes sense in general metric spaces, but does not have the same implications.

---

4. **Vector spaces with inner products**

Let \( V \) be a vector space (over \( \mathbb{R} \) or \( \mathbb{C} \)). An inner product or scalar product or dot product on \( V \) is a \( \mathbb{C} \)-valued function \((,): V \times V \to \mathbb{C}\) which is linear in its first argument:

\[
\langle ax + bx', y \rangle = a\langle x, y \rangle + b\langle x', y \rangle 
\]

hermitian in the sense that

\[
\langle y, x \rangle = \overline{\langle x, y \rangle}
\]

and positive in the sense that \( \langle x, x \rangle \geq 0 \), with equality only for \( x = 0 \). This entails that the inner product is conjugate-linear in its second argument:

\[
\langle x, ay + by' \rangle = \overline{a}\langle x, y \rangle + \overline{b}\langle x, y' \rangle
\]

(for scalars \( a, b \) and \( x, x', y \in V \))

We say that \( V \) is an inner-product space or pre-Hilbert space.

An inner product gives rise to a norm \( |x| = \langle x, x \rangle^{1/2} \), and a metric by \( d(x, y) = |x - y| \). The norm is homogeneous in the sense that

\[
|cx| = \langle cx, cx \rangle^{1/2} = (c\overline{c} \cdot \langle x, x \rangle)^{1/2} = |c|\cdot |x|
\]

and positive in the sense that \( |x| \geq 0 \), with \( |x| = 0 \) only for \( x = 0 \).

The proof of Cauchy-Schwarz-Bunyakowsky given earlier in the context of \( \mathbb{R}^n \) applies in arbitrary inner-product spaces (with minor technical accommodation of complex scalars):

\[
|\langle x, y \rangle| \leq |x| \cdot |y| 
\]

(for all \( x, y \in V \))
with equality only for \( x, y \) collinear. Then the triangle inequality for this norm follows, as well:
\[
|x + y|^2 = |x|^2 + \langle x, y \rangle + \langle y, x \rangle + |y|^2 = |x|^2 + 2\text{Re}\langle x, y \rangle + |y|^2 \leq |x|^2 + 2|x, y| + |y|^2
\]
\[
\leq |x|^2 + 2|x||y| + |y|^2 = (|x| + |y|)^2
\]
When \( V \) is complete with respect to the metric, \( V \) is called a Hilbert space. Finite-dimensional inner-product spaces are always complete. The simplest infinite-dimensional Hilbert space is \( \ell^2 \). The completeness requires proof.

5. Vector spaces with norms

Let \( V \) be a vector space (over \( \mathbb{R} \) or \( \mathbb{C} \)). A norm on \( V \) is an \( \mathbb{R} \)-valued function \( |\cdot| : V \to \mathbb{R} \) with homogeneity
\[
|cx| = |c| |x|_V \quad \text{for } c \in \mathbb{C} \text{ and } x \in V
\]
satisfying the triangle inequality
\[
|x + y| \leq |x| + |y|
\]
and and positive in the sense that \( |x| \geq 0 \), with equality only for \( x = 0 \). We may say that \( V \) is a normed space or pre-Banach space.

When \( V \) is complete with respect to this metric, it is called a Banach space. Finite-dimensional normed spaces are always complete. A simple infinite-dimensional Banach space is \( C^0[a, b] \). The completeness requires proof.

The normed spaces \( \ell^p \) with \( 1 \leq p < \infty \) are complete, so are Banach spaces. Proving that the triangle inequality is met requires proof, as does the completeness.

Most norms do not arise from inner products. If \( |x| = \langle x, x \rangle^{\frac{1}{2}} \) does arise from an inner product \( \langle , \rangle \), then the inner product can be recovered from the norm via the polarization/parallelogram identity: for simplicity, suppose the scalars are \( \mathbb{R} \) rather than \( \mathbb{C} \), to dodge complex conjugation for a moment:
\[
|x + y|^2 - |x - y|^2 = \langle x + y, x + y \rangle - \langle x - y, x - y \rangle = (\langle x, x \rangle + \langle x, y \rangle + \langle y, x \rangle + \langle y, y \rangle) - (\langle x, x \rangle - \langle x, y \rangle - \langle y, x \rangle + \langle y, y \rangle) = 4\langle x, y \rangle
\]
For complex scalars, we use \( x \pm iy \) and \( x \pm iy \), in a slightly more complicated identity, to recover the inner product from the norm.

In particular, if the function \( |x + y|^2 - |x - y|^2 \) for a given norm does not have the properties of an inner product, then that norm cannot arise from any inner product.

Metrics on vector spaces arising from norms are translation-invariant:
\[
d(x + z, y + z) = |(x + z) - (y + z)| = |x - y| = d(x, y)
\]
and homogeneous:
\[
d(cx, cy) = |cx - cy| = |c| |x - y|_V = |c| \cdot d(x, y)
\]

6. Product topologies and metrics

The product topology on the Cartesian product \( \prod_{\alpha \in A} X_\alpha \) for topological spaces \( X_\alpha \) is usually described as having sub-basis of sets \( \prod_{\alpha \in A} U_\alpha \) where for all indices except a single one \( \alpha_0 \), we have \( U_\alpha = X_\alpha \), and for that single \( \alpha_0 \) the set \( U_{\alpha_0} \) is any open set in \( X_{\alpha_0} \).
For infinite index sets $A$, this topology is coarser than may be expected. In particular, for infinite $A$, the product topology is usually not the box topology on $\prod_{\alpha} X_{\alpha}$, which has basis of sets $\prod_{\alpha} U_{\alpha}$ where each $U_{\alpha}$ is an open in $X_{\alpha}$.

An appendix explains the coarseness of the product topology, and, in particular, why it is correct, and why it is uniquely determined.

[6.1] Theorem: Countable products of metric spaces are metric spaces. If all the factors are complete, then the product is complete.

Proof: Let $X = \prod_{\alpha} X_{\alpha}$, where $X_{\alpha}$ has metric $d_{\alpha}$. On $X$ consider the function

$$d((x_1, x_2, \ldots), (y_1, y_2, \ldots)) = \sum_{n} 2^{-n} \frac{d_{\alpha}(x_{n}, y_{n})}{1 + d_{\alpha}(x_{n}, y_{n})} \quad \text{(for } x_{n}, y_{n} \in X_{n} \text{ for } n = 1, 2, \ldots)$$

One trick here is that $\frac{t}{1+t} \leq 1$ for all non-negative reals $t$, so the sum converges. In fact, there are many other similar expressions that achieve the same effect, and give the same topology.

There is some work to be done to prove that this expression gives a metric, and that the topology attached to that metric is the product topology...

7. Topological vector spaces

There are some important, translation-invariant metrics on vector spaces which do not come from norms by the recipe $d(x, y) = |x - y|$. In fact, noting that many different metrics can give the same topology, there are some topologies on vector spaces which cannot come from norms.

More precisely, a topological vector space $V$ is a (probably real or complex) vector space with a topology, such that scalar multiplication and vector addition are continuous, and that points are closed sets. (The latter condition excludes some pathologies, and in fact implies that the topology is Hausdorff.) Inner product spaces and normed vector spaces all fit into this category.

[7.1] Claim: A topological vector space topology on $V$ is translation-invariant in the sense that for open $U \subset V$, and for $v \in V$ the translate $U + v$ is open.

Proof: Fix $v \in V$. The addition and subtraction maps $A_v, S_v$ defined by $A_v(w) = w + v$ and $S_v(w) = w - v$ are continuous, by assumption, and are mutual inverses. Thus, they are both homeomorphisms. Thus, $A_v(U) = U + v$ is open.

[7.2] Corollary: To describe the topology on a topological vector space, it suffices to give a local basis at 0.

The fundamental example of a vector space whose natural topology can be given by a complete, translation-invariant metric, but whose topology cannot arise from a norm, is

$$C^\infty[a, b] = \bigcap_{k \geq 0} C^k[a, b]$$

where each space $C^k[a, b]$ of $k$-fold continuously differentiable functions does have (complete, metric) topology given by the norm

$$|f|_{C^k} = \sum_{i=0}^{k} |f^{(i)}|_{C^0}$$
The completeness of each $C^k[a,b]$ does require some proof, and for $k > 0$ seems to need the fundamental theorem of calculus. The usual metric on $C^\infty[a,b]$ is

$$d(f,g) = \sum_{n \geq 0} 2^{-n} \frac{|f^{(n)}(a) - g^{(n)}(a)|_{C^0} + |f^{(n)}(b) - g^{(n)}(b)|_{C^0}}{1 + |f^{(n)}(c) - g^{(n)}(c)|_{C^0}}$$

The completeness requires some work, similar to proving that a countable product of complete metric spaces is complete. Thus, $C^\infty[a,b]$ is a Fréchet space.\[6\] Much as there is an argument for the correctness of the product topology, there is an argument for the correctness of this topology on $C^\infty[a,b]$. The metric itself is non-unique, but the correct topology is unique. The following example shows that the class of Banach spaces is not broad enough.

[7.3] Claim: There is no norm on $C^\infty[a,b]$ giving the natural (complete metric) topology above.

Proof: [...] iou [...]

[7.4] Remark: Also, as remarked earlier, the correct topology on the space $C^\infty(\mathbb{R})$ of test functions cannot be given by any metric. Likewise weak dual topologies are rarely given by metrics.

### 8. Completions of metric spaces

Again, a metric space $X,d$ is (sequentially) complete when every Cauchy sequence is convergent. Completeness is a convenient feature, because then we can take limits without leaving the space. As in the example of $C^0[a,b]$ with the $L^2$ inner product, we might want to imbed a non-complete metric space in a complete one in an optimal and universal way.

A traditional notion of the completion of a metric space $X$ is a construction of a complete metric space $\bar{X}$ with a distance-preserving injection $j : X \to \bar{X}$ so that $j(X)$ is dense in $\bar{X}$, in the sense that every point of $\bar{X}$ is the limit of a Cauchy sequence in $j(X)$.

The intention is that every Cauchy sequence has a limit, so we should (somehow!) adjoin points as needed for these limits. However, different Cauchy sequences may happen to have the same limit.

Thus, we want an equivalence relation on Cauchy sequences that says they should have the same limit, even without knowing the limit exists or having somehow constructed or adjoined the limit point.

Define an equivalence relation $\sim$ on the set $C$ of Cauchy sequences in $X$, by

$$\{x_s\} \sim \{y_t\} \iff \lim_{s} d(x_s, y_s) = 0$$

Attempt to define a metric on the set $C/\sim$ of equivalence classes by

$$d(\{x_s\}, \{y_t\}) = \lim_{s} d(x_s, y_s)$$

We must verify that this is well-defined on the quotient $C/\sim$ and gives a metric. We have an injection $j : X \to C/\sim$ by

$$x \to \{x, x, x, \ldots\} \mod \sim$$

[8.1] Claim: $j : X \to C/\sim$ is a completion of $X$.

\[6\] There is another requirement for a vector space to be a Fréchet space, namely, that open balls are convex. This local convexity requirement can be checked for $C^\infty[a,b]$.
Proof: Grant for the moment that the distance function on $\bar{X} = C/\sim$ is well-defined, and is complete, and show that it has the property of a completion of $X$. To this end, let $f : X \to Y$ be a uniformly continuous map to a complete metric space $Y$.

Given $z \in \bar{X}$, choose a Cauchy sequence $x_k$ in $X$ with $j(x_k)$ converging to $z$, and try to define $F : \bar{X} \to Y$ in the natural way, by

$$F(z) = \lim_k f(x_k)$$

Since $f$ is uniformly continuous, $f(x_k)$ is Cauchy in $Y$, and by completeness of $Y$ has a limit, so $F(z)$ exists, at least if well-defined.

For well-definedness of $F(z)$, for $x_k$ and $x'_k$ two Cauchy sequences whose images $j(x_k)$ and $j(x'_k)$ approach $z$, since $j$ is an isometry eventually $x_k$ is close to $x'_k$, so $f(x_k)$ is eventually close to $f(x'_k)$ in $Y$, showing $F(z)$ is well-defined.

We saw that every element of $\bar{X}$ is a limit of a Cauchy sequence $j(x_k)$ for $x_k$ in $X$, and any continuous $\bar{X} \to Y$ respects limits, so $F$ is the only possible extension of $f$ to $\bar{X}$.

The obvious argument will show that $F$ is continuous. Namely, let $z, z' \in \bar{X}$, with Cauchy sequences $x_t$ and $x'_t$ approaching $z$ and $z'$. Given $\varepsilon > 0$, by uniform continuity of $F$, there is $N$ large enough such that $d_Y(F(j(x_r)), F(j(x_s))) < \varepsilon$ and $d_Y(F(j(x'_r)), F(j(x'_s))) < \varepsilon$ for $r,s \geq N$. From the lemma above (!), for such $r$ even in the limit the strict inequalities are at worst non-strict inequalities:

$$d_Y(f(x_r), F(z)) \leq \varepsilon \quad \text{and} \quad d_Y(f(x'_r), F(z')) \leq \varepsilon$$

By the triangle inequality, since $f : X \to Y$ is continuous, we can increase $r$ to have $d_X(x_r, x'_r)$ small enough so that $d_Y(f(x_r), f(x'_r)) < \varepsilon$, and then

$$d_Y(F(z), F(z')) \leq d_Y(F(z), f(x_r)) + d_Y(f(x_r), f(x'_r)) + d_Y(f(x'_r), F(z')) \leq \varepsilon + \varepsilon + \varepsilon$$

Since $j : X \to \bar{X}$ is an isometry,

$$d_X(x_r, x'_r) = d_{\bar{X}}(j(x_r), j(x'_r)) \leq d_{\bar{X}}(j(x_r), z) + d_{\bar{X}}(z, z') + d_{\bar{X}}(j(x'_r), z')$$

so

$$d_X(x_r, x'_r) \leq d_{\bar{X}}(z, z') + 2\varepsilon$$

Thus,

$$d_Y(F(z), F(z')) \leq d_{\bar{X}}(z, z') + 4\varepsilon \quad \text{(for all } \varepsilon > 0)$$

Thus, $F$ is continuous. Granting that $\bar{X} = C/\sim$ is complete, etc., it is a completion of $X$.

It remains to prove that the apparent metric on $\bar{X}$ truly is a metric, and that $\bar{X}$ is complete.

First, the limit in attempted definition

$$d(\{x_i\}, \{y_i\}) = \lim_s d(x_s, y_s)$$

does exist: given $\varepsilon > 0$, take $N$ large enough so that $d(x_i, x_j) < \varepsilon$ and $d(y_i, y_j) < \varepsilon$ for $i,j \geq N$. By the triangle inequality,

$$d(x_i, y_i) \leq d(x_i, x_N) + d(x_N, y_N) + d(y_N, y_i) < \varepsilon + d(x_N, y_N) + \varepsilon$$

Similarly,

$$d(x_i, y_i) \geq -d(x_i, x_N) + d(x_N, y_N) - d(y_N, y_i) > -\varepsilon + d(x_N, y_N) - \varepsilon$$

Thus, unsurprisingly,

$$|d(x_i, y_i) - d(x_N, y_N)| < 2\varepsilon$$
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and the sequence of real numbers \( d(x_i, y_i) \) is Cauchy, so convergent.

Similarly, when \( \lim_i d(x_i, y_i) = 0 \), then \( \lim_i d(x_i, z_i) = \lim_i d(y_i, z_i) \) for any other Cauchy sequence \( z_i \), so the distance function is well-defined on \( C/\sim \).

The positivity and symmetry for the alleged metric on \( C/\sim \) are immediate. For triangle inequality, given \( x_i, y_i, z_i \) and \( \varepsilon > 0 \), let \( N \) be large enough so that \( d(x_i, x_j) < \varepsilon , d(y_i, y_j) < \varepsilon \), and \( d(z_i, z_j) < \varepsilon \) for \( i, j \geq N \).

As just above,
\[
\left| d(\{x_s\}, \{y_s\}) - d(x_i, y_i) \right| < 2\varepsilon
\]

Thus,
\[
d(\{x_s\}, \{y_s\}) \leq 2\varepsilon + d(x_N, y_N) \leq 2\varepsilon + d(x_N, z_N) + d(z_N, y_N) \leq 2\varepsilon + d(\{x_s\}, \{z_s\}) + 2\varepsilon + d(\{z_s\}, \{y_s\}) + 2\varepsilon
\]

This holds for all \( \varepsilon > 0 \), so we have the triangle inequality.

Finally, perhaps anticlimactically, the completeness. Given Cauchy sequences \( c_s = \{x_s\} \) in \( X \) such that \( \{c_s\} \) is Cauchy in \( C/\sim \), for each \( s \) we will choose large-enough \( j(s) \) such that the diagonal-ish sequence \( y_s = x_{t,j(t)} \) is a Cauchy sequence in \( X \) to which \( \{c_s\} \) converges.

Given \( \varepsilon > 0 \), take \( i \) large enough so that \( d(c_s, c_t) < \varepsilon \) for all \( s, t \geq i \). For each \( i \), choose \( j(i) \) large enough so that \( d(x_{ij}, x_{ij'}) < \varepsilon \) for all \( j, j' \geq j(i) \). Let \( c = \{x_{t,i(t)} : i = 1, 2, \ldots \} \). For \( s \geq i \),
\[
d(c_s, c) = \lim_{\ell} d(x_{s,\ell}, x_{t,j(\ell)}) \leq \sup_{\ell \geq 1} d(x_{s,\ell}, x_{t,j(\ell)}) \leq \sup_{\ell \geq 1} \left( d(x_{s,\ell}, x_{s,j(\ell)}) + d(x_{s,j(\ell)}, x_{t,j(\ell)}) \right) \leq 2\varepsilon
\]

This holds for all \( \varepsilon > 0 \), so \( \lim_s c_s = c \), and \( C/\sim \) is complete. ///

Many natural metric spaces are complete without any need to complete them. The historically notable exception was \( \mathbb{Q} \) itself, completed to \( \mathbb{R} \). A slightly more recent example:

[8.2] Example: One description of the space \( L^2[a,b] \) is as the completion of \( C^0[a,b] \) with respect to the \( L^2 \) norm above. The more common description depends on notions of measurable function and Lebesgue integral, and presents the space as equivalence classes of functions, having somewhat ambiguous pointwise values.

9. Topologies more general than metric topologies

Many of the ideas and bits of terminology for metric spaces make sense and usefully extend to more general situations. Some do not.

[9.1] A topology on a set \( X \) is a collection \( \tau \) of subsets of \( X \), called the open sets, such that \( X \) itself and the empty set \( \emptyset \) are in \( \tau \), arbitrary unions of elements of \( \tau \) are in \( \tau \), and finite intersections of elements of \( \tau \) are in \( \tau \). A set \( X \) with an explicitly or implicitly specified topology is a topological space.

[9.2] Finer/coarser, stronger/weaker topologies From thinking of metric spaces, the relevance of the following ideas/definitions is surely not clear. Nevertheless, eventually, it is convenient to have names for these phenomena.

Given a set \( X \) and topologies \( \sigma, \tau \) (subsets of the power set of \( X \)) on \( X \), we say that \( \sigma \) is weaker or coarser than \( \tau \) when \( \sigma \subset \tau \). And \( \tau \) is stronger or finer than \( \sigma \).

That is, topology \( \sigma \) is weaker/coarser when every open of \( \sigma \) is an open of \( \tau \). Equivalently, \( \tau \) is stronger/finer.

[9.3] A continuous map \( f : X \rightarrow Y \) for topological spaces \( X, Y \) is a set-map so that inverse images \( f^{-1}(U) \) of opens \( U \) in \( Y \) are open in \( X \).
Uniform continuity of functions or maps has no natural formulation in general topological spaces, in effect because we have no device by which to compare the topology at varying points, unlike the case of metric spaces, where there is a common notion of distance that does allow such comparisons.

**[9.4] Closed sets** in a topological space are exactly the complements of open sets. Arbitrary intersections of closed sets are closed, and finite unions of closed sets are closed.

**[9.5] A basis for a topology** is a collection of (open) subsets so that any open set is a union of the (open) sets in the basis. In a metric space, the open balls of all possible sizes, at all points, are a natural basis.

**[9.6] A neighborhood of a point** is any set containing an open set containing the point. Often, one considers only open neighborhoods, to avoid irrelevant misunderstandings.

**[9.7] A local basis** at a point $x$ in a space $X$ is a collection of open neighborhoods of $x$ such that every neighborhood of $x$ contains a neighborhood from the collection. In a metric space, the collection of open balls at a given point with rational radius is a countable local basis at that point.

**[9.8] The closure of a set** $E$ (in a topological space $X$), sometimes denoted $\overline{E}$, is the intersection of all closed sets containing $E$. It is a closed set. Equivalently, it is the set of $x \in X$ such that every neighborhood of $x$ meets $E$. The closure of $E$ contains $E$.

**[9.9] The interior of a set** $E$ (in a topological space $X$) is the union of all open sets contained in it. It is open. Equivalently, it is the set of $x \in X$ such that there is a neighborhood of $x$ inside $E$. The interior of $E$ is a subset of $E$.

**[9.10] The boundary of a set** $E$ (in a topological space $X$), often denoted $\partial E$, is the intersection of the closure of $E$ and the closure of the complement of $E$. Equivalently, it is the set of $x \in X$ such that every neighborhood of $x$ meets both $E$ and the complement of $E$.

**[9.11] A Hausdorff topology** is one in which any two points $x, y$ have neighborhoods $U \ni x$ and $V \ni y$ which are disjoint: $U \cap V = \emptyset$. This is a reasonable condition to impose on a space on which functions should live.

**[9.12] Claim:** Metric spaces are Hausdorff.

*Proof:* Given $x \neq y$ in a metric space, let $B_1$ be the open ball of radius $d(x, y)/2$, and let $B_2$ the open ball of radius $d(x, y)/2$ at $y$. For any $z \in B_1 \cap B_2$, by the triangle inequality,

$$d(x, y) \leq d(x, z) + d(z, y) < \frac{d(x, y)}{2} + \frac{d(x, y)}{2} = d(x, y)$$

which is impossible. Thus, there is no $z$ in the intersection of these two open neighborhoods of $x$ and $y$.

///

**[9.13] Claim:** In Hausdorff spaces, singleton sets $\{x\}$ are closed.

*Proof:* Fixing $x$, for $y \neq x$ let $U_y$ be an open neighborhood of $y$ not containing $x$. (We do not use the open neighborhood of $x$ not meeting $U_y$.) Then $E = \bigcup_{y \neq x} U_y$ is open, does not contain $x$, and contains every other point in the space. Thus, $E$ is the complement of the singleton set $\{x\}$ and is open, so $\{x\}$ is closed.

///

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[7] A set $X$ *meets* another set $Y$ if $X \cap Y \neq \emptyset$. 

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[9.14] Convergence of sequences: In a topological space $X$, a sequence $x_1, x_2, \ldots$ converges to $x_\infty \in X$, written $\lim_n x_n = x_\infty$, if, for every neighborhood $U$ of $x_\infty$, there is an index $m$ such that for all $n \geq m$, $x_n \in U$.

In more general, non-Hausdorff spaces, it is easily possible to have a sequence converge to more than one point, which is fairly contrary to our intention for the notion of convergence.

In a metric space, the notion of Cauchy sequence has a sense, and in a complete metric space, the notions of Cauchy sequence and convergent sequence are identical, and there is a unique limit to which such a sequence converges.

In more general, non-Hausdorff spaces, and not-locally-countably-based spaces, things can go haywire in several different ways, which are mostly irrelevant to the situations we care about. Still, one should be aware that not all spaces are Hausdorff, and may fail to be countably locally based.

[9.15] Sequentially compact sets $E$ in a topological space $X$ are those such that every sequence has a convergent subsequence (with limit in $E$).

Although the definition of convergent does not directly mention potential difficulties and ambiguities, there are indeed problems in non-Hausdorff spaces, and in spaces that fail to have countable local bases.

[9.16] Accumulation points of a subset $E$ of a topological space $X$ are points $x \in X$ such that every neighborhood of $x$ contains infinitely-many elements of $E$. Every accumulation point of $E$ lies in the closure of $E$, but not vice-versa.

[9.17] Claim: A closed set $E$ is sequentially compact if and only if every sequence in $E$ either has an accumulation point in $E$, or contains only finitely-many distinct points.

Proof: First, the technicality: if a sequence contains only finitely-many distinct points, it cannot have any accumulation points, but certainly contains convergent subsequences. For a sequence $x_1, x_2, \ldots$ including infinitely-many distinct points, drop any repeated points, so that $x_i \neq x_j$ for all $i \neq j$. For $E$ sequentially compact, there is a subsequence with limit $x_\infty$ in $E$. Relabel if necessary so that the subsequence is still denoted $x_1, x_2, \ldots$. The subsequence still consists of mutually distinct points. Since $\lim_n x_n = x_\infty$, given a neighborhood $U$ of $x_\infty$, there is $m$ such that $x_n \in U$ for all $n \geq m$. Since $x_m, x_{m+1}, \ldots$ is an infinite set of distinct points, $x_\infty$ is an accumulation point of the subsequence, hence, of the original sequence.

Conversely, if a sequence has an accumulation point, it has a subsequence converging to that accumulation point. ///

[9.18] Compact sets in topological spaces are subsets such that every open cover has a finite subcover. That is, $K$ is compact when, for any collection of open sets $\{U_{\alpha} : \alpha \in A\}$ such that $K \subset \bigcup_{\alpha \in A} U_{\alpha}$, there is a finite collection $U_{\alpha_1}, \ldots, U_{\alpha_n}$ such that $K \subset U_{\alpha_1} \cup \ldots \cup U_{\alpha_n}$.

[9.19] Claim: For $f : X \to Y$ continuous and $K$ compact in $X$, the image $f(K)$ is compact in $Y$.

Proof: Given an open cover $\{U_{\alpha} : \alpha \in A\}$ of $f(K)$, the inverse images $f^{-1}(U_{\alpha})$ give an open cover of $K$. Thus, there is a finite subcover $f^{-1}(U_{\alpha_1}), \ldots, f^{-1}(U_{\alpha_n})$. Then $U_{\alpha_1}, \ldots, U_{\alpha_n}$ is a (finite) cover of $f(K)$.

Since singleton sets $\{x\}$ are certainly compact, the following generalizes the earlier claim about closedness of singleton sets in Hausdorff spaces:


Proof: Let $E$ be a compact subset of $X$. For $y \notin E$, for each $x \in E$, let $U_x \ni y$ be open and $V_x \ni x$ open
so that $U_x \cap V_x = \emptyset$. Then $\{V_x : x \in E\}$ is an open cover of $E$, with finite subcover $E \subset V_{x_1} \cup \ldots \cup V_{x_n}$. The finite intersection $W_y = U_{x_1} \cap \ldots \cap U_{x_n}$ is open, and disjoint from $V_{x_1} \cup \ldots \cup V_{x_n}$, so is disjoint from $E$. Thus, $W_y$ is open and contains $y$. The union $W = \bigcup_{y \notin E} W_y$ is open, and contains every $y \notin E$. Thus $E$ is the complement of an open set, so is closed.


Proof: Let $X$ be the ambient space, and $K_\alpha$ the compacts, with index set $A$ totally ordered, in the sense $A$ has an order relation $<$ such that for every distinct $\alpha, \beta \in A$, either $\alpha < \beta$ or $\beta < \alpha$. The nested condition is that if $\alpha < \beta$ then $K_\alpha \supset K_\beta$. (It can equally well be the opposite direction of containment.) We claim that $\bigcap_\alpha K_\alpha$ is compact.

From above, each $K_\alpha$ is closed, so the complements $U_\alpha = X - K_\alpha$ are open. If $\bigcap_\alpha K_\alpha = \emptyset$, then $\bigcup_\alpha U_\alpha = X$. In particular, $\bigcup_\alpha U_\alpha \supset K_\beta$ for all indices $\beta$. For fixed index $\alpha_\beta$, let $U_{\alpha_1} \cup \ldots \cup U_{\alpha_n}$ be a finite subcover of $K_\alpha$, so certainly a cover of $K_{\alpha'}$ for all $\alpha' > \alpha$. Because of the nested-ness, for $\beta = \max\{\alpha_1, \ldots, \alpha_n\}$, $U_\beta = U_{\alpha_1} \cup \ldots \cup U_{\alpha_n}$. But $U_\beta$ is the complement of $K_\beta$, so certainly cannot cover it, contradiction. ///

[9.22] A locally compact topology is one in which every point has a neighborhood with compact closure. This is a reasonable condition to impose on a space on which functions will live. $\mathbb{R}^n$ is locally compact, but the metric space $\ell^2$ is not. Later, we will see that no infinite-dimensional Hilbert space or Banach space is locally compact. That is, natural spaces of functions are not usually locally compact, but the physical spaces on which the functions live usually are locally compact.

[9.23] Separable topological spaces are those with countable dense subsets. For example, the countable set $\mathbb{Q}^n$ is dense in $\mathbb{R}^n$. Nearly all topological spaces arising in practice are separable, but most basic results do not directly use this property.

[9.24] Countably-based topological spaces are those with a countable basis. Sometimes such spaces are called second-countable. Perhaps counter-intuitively, first-countable spaces are those in which every point has a countable local basis. Many topological spaces arising in practice are countably-based, but most basic results do not directly use this property.

[9.25] Claim: Separable metric spaces are countably-based. Specifically, for countable dense subset $S$ of metric space $X$, open balls of rational radius centered at points of $S$ form a basis.

Proof: Since there are only countably-many $s \in S$ and only countably many rational radiuses, the set of such open balls is indeed countable.

Fix an open $U \subset X$. Given $x \in U$, let $r > 0$ be sufficiently small so that the open ball at $x$ of radius $r$ is inside $U$. Let $s_x \in S$ be such that $d(x, s) < r/2$. By density of rational numbers in $\mathbb{R}$, there is a rational number $q_x$ such that $d(x, s) < q_x < r/2$. Thus, by the triangle inequality, the ball $B_x$ at $s_x$ of radius $q_x$ contains $x$ and lies inside the open ball at $x$ of radius $r$, so $B_x \subset U$.

The union of all $B_x$ over $x \in U$ is a subset of $U$ containing all $x \in U$, so is $U$ itself. ///

[9.26] Test functions $C^\infty(\mathbb{R}^n)$. The space $C^\infty(\mathbb{R}^n)$ of test functions on $\mathbb{R}^n$ is the space of smooth, compactly-supported (complex-valued) functions. As usual, we would like to be able to take suitable limits and stay in side this space. The question is, what limits are suitable? That is, what topology makes the space of test functions (suitably) complete? It turns out that the appropriate complete topology is not metric, because any remotely reasonable topology on $C^\infty(\mathbb{R}^n)$ violates the conclusion of the Baire category theorem (below). This important example illustrates the practical need for topologies more general than metric topologies. This topology will be considered in detail later.
[9.27] **Weak topologies** Another type of topology that is not metric is the *weak dual* or *weak* * topology on the *dual space*

\[ V^* = \{ \text{continuous linear maps } V \to \mathbb{C} \} \]

for vector spaces \( V \) with topologies. This sort of weak topology is much *coarser* than the Hilbert-space dual topology on duals of Hilbert spaces, and much *coarser* than the Banach space dual topology on the duals of Banach spaces. Weak dual topologies will be considered in detail later.

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**10. Compactness versus sequential compactness**

In general topological spaces, *compactness* is a stronger condition than *sequential compactness*. First, without any further hypotheses on the spaces, however noting the point that sequential compactness easily fails to be what we anticipate in topological spaces that are not necessarily Hausdorff or locally countably-based:

**[10.1] Claim:** Compact sets are sequentially compact.

*Proof:* Given a sequence, if some \( y \in E \) is an accumulation point, then there is a subsequence converging to \( y \), and we are done. If no \( y \in E \) is an accumulation point of the given sequence, then each \( y \in E \) has an open neighborhood \( U_y \) such that \( U_y \) meets the sequence in only finitely-many points. The sets \( U_y \) cover \( E \). For \( E \) compact, there is a finite subcover \( U_{y_1}, \ldots, U_{y_n} \). Each \( U_y \) contains only finitely-many points of the sequence, so the sequence contains only finitely-many distinct points, so certainly has a convergent subsequence. ///

**[10.2] Claim:** In a countably-based topological space \( X \), sequentially compact sets are compact.

*Proof:* Let \( E \subset X \) be sequentially compact. The opens in an arbitrary cover of \( E \) are (necessarily countable) unions of some of the countably-many opens in the countable basis for \( X \). Thus, it suffices to show that a *countable* cover \( E \subset U_1 \cup U_2 \cup \ldots \) admits a finite subcover.

If no finite collection of the \( U_n \) covers \( E \), then for each \( n = 1, 2, \ldots \) there is \( e_n \in E \) such that \( e_n \notin U_1 \cup \ldots \cup U_n \). Since every \( e_n \) does lie in some \( U_k \), we can replace \( \{e_n\} \) by a subsequence so that \( e_i \neq e_j \) for all \( i \neq j \), and still \( e_n \notin U_1 \cup \ldots \cup U_n \).

By sequential compactness, \( e_1, e_2, \ldots \) has a convergent subsequence, with limit \( e_\infty \in E \). The point \( e_\infty \) lies in some \( U_m \). Thus, there would be infinitely-many indices \( n \) such that \( e_n \in U_m \). This is impossible, since \( e_n \notin U_1 \cup \ldots \cup U_n \). Thus, there must be a finite subcover. ///

The argument for the previous claim can be improved, to show

**[10.3] Claim:** In complete metric spaces, sequentially compact sets are compact.

*Proof:* Let \( \{U_\alpha : \alpha \in A\} \) be an open cover of a subset \( E \) of a complete metric space \( X \), admitting no finite subcover. Using an equivalent of the Axiom of Choice, we can arrange to have a *minimal* subcover, that is, so that no \( U_\beta \) can be removed an still cover \( E \). We do this at the end of the argument.

Granting this, without loss of generality the open cover is *minimal*, and not finite. Using the minimality (and again using the Axiom of Choice), for each index \( \beta \in A \), let \( x_\beta \) be a point in \( E \) that is *not* in \( \bigcup_{\alpha \neq \beta} U_\alpha \). Since the cover is minimal, these \( x_\beta \)'s must be *distinct*. Since the cover is not finite, there are infinitely-many (distinct) \( x_\beta \)'s. Since the are distinct, any countable subset of \( \{x_\beta : \beta \in A\} \) gives a sequence \( y_1, y_2, \ldots \) of distinct points. By sequential compactness, this sequence has at least one accumulation point \( y_\infty \in E \).

Let \( U_\alpha \) be an open in the cover containing \( y_\infty \). Since \( \lim n y_n = y_\infty \), there is \( n_o \) such that for all \( n \geq n_o \) we have \( y_n \in U_\alpha \). All those \( y_n \)'s are among the \( x_\beta \)'s, but the only \( x_\beta \) in \( U_\alpha \) is \( x_\alpha \). That is, there cannot be infinitely-many distinct \( x_\beta \)'s in \( U_\alpha \). Thus, assuming that a minimal cover is infinite leads to a contradiction.
To obtain a minimal subcover from a given cover \( \{ U_\alpha : \alpha \in A \} \), well-order the index set \( A \). We choose a minimal subcover by transfinite induction, as follows. The idea is to ask, in the order chosen for \( A \), cumulatively, whether or not \( U_\alpha \) can be removed from the current subcover while still having a cover of the given set. That is, we inductively define a subset \( B \) of the index set \( A \) by transfinite induction: initially, \( B = A \). At the \( \alpha \)th stage, remove \( \alpha \) from \( B \) if \( U_\alpha \) is unnecessary for maintaining the cover property. That is, remove \( \alpha \) if
\[
E \subset \bigcup_{\beta < \alpha, \beta \in B} U_\beta \cup \bigcup_{\beta > \alpha} U_\beta
\]
only Otherwise keep \( \alpha \) in \( B \). By transfinite induction, \( B \) is an index set for a subcover of \( \{ U_\alpha : \alpha \in A \} \), and that subcover is minimal in the sense that no open can be removed without the result failing to be a cover.

11. **Total boundedness criterion for pre-compactness**

In general metric spaces, closed and bounded sets need not be compact (nor sequentially compact). For example, the closed unit ball
\[
B = \{ v \in B : \left( \sum_n |v_n|^2 \right)^{\frac{1}{2}} \leq 1 \} \subset \ell^2
\]
is not sequentially compact: the vectors
\[
e_1 = (1, 0, 0, \ldots)
e_2 = (0, 1, 0, \ldots)
e_3 = (0, 0, 1, 0, \ldots)
\]
... have distance \( \sqrt{2} \) from each other, so cannot have a convergent subsequence, since elements of a convergent (sub-) sequence get closer and closer together. More is required, as follows.

A set \( E \) in a metric space is totally bounded if, given \( \varepsilon > 0 \), there are finitely-many open balls of radius \( \varepsilon \) covering \( E \). The property of total boundedness in a metric space is generally stronger than mere boundedness. It is immediate that any subset of a totally bounded set is totally bounded.

[11.1] **Theorem:** A set \( E \) in a metric space \( X \) has compact closure if and only if it is totally bounded.

[11.2] **Remark:** Sometimes a set with compact closure is said to be pre-compact.

**Proof:** Certainly if a set has compact closure then it admits a finite covering by open balls of arbitrarily small (positive) radius, by the compactness.

On the other hand, suppose that a set \( E \) is totally bounded in a complete metric space \( X \). To show that \( E \) has compact closure it suffices to show sequential compactness, namely, that any sequence \( \{ x_i \} \) in \( E \) has a convergent subsequence.

We choose such a subsequence as follows. Cover \( E \) by finitely-many open balls of radius 1, invoking the total boundedness. In at least one of these balls there are infinitely-many elements from the sequence. Pick such a ball \( B_1 \), and let \( i_1 \) be the smallest index so that \( x_{i_1} \) lies in this ball.

The set \( E \cap B_1 \) is still totally bounded (and contains infinitely-many elements from the sequence). Cover it by finitely-many open balls of radius \( 1/2 \), and choose a ball \( B_2 \) with infinitely-many elements of the sequence lying in \( E \cap B_1 \cap B_2 \). Choose the index \( i_2 \) to be the smallest one so that both \( i_2 > i_1 \) and so that \( x_{i_2} \) lies inside \( E \cap B_1 \cap B_2 \).

Proceeding inductively, suppose that indices \( i_1 < \ldots < i_n \) have been chosen, and balls \( B_i \) of radius \( 1/i \), so that
\[
x_i \in E \cap B_1 \cap B_2 \cap \ldots \cap B_i
\]
Then cover $E \cap B_1 \cap \ldots \cap B_n$ by finitely-many balls of radius $1/(n+1)$ and choose one, call it $B_{n+1}$, containing infinitely-many elements of the sequence. Let $i_{n+1}$ be the first index so that $i_{n+1} > i_n$ and so that

$$x_{n+1} \in E \cap B_1 \cap \ldots \cap B_{n+1}$$

Then for $m < n$ we have $d(x_{i_m}, x_{i_n}) \leq \frac{1}{m}$ so this subsequence is Cauchy. \///

12. Baire’s theorem

This standard result is both indispensable and mysterious.

A set $E$ in a topological space $X$ is nowhere dense if its closure $\overline{E}$ contains no non-empty open set. A countable union of nowhere dense sets is said to be of first category, while every other subset (if any) is of second category. The idea (not at all clear from this traditional terminology) is that first category sets are small, while second category sets are large. In this terminology, the theorem’s assertion is equivalent to the assertion that (non-empty) complete metric spaces and locally compact Hausdorff spaces are of second category.

A $G_\delta$ set is a countable intersection of open sets. Concommitantly, an $F_\sigma$ set is a countable union of closed sets. Again, the following theorem can be paraphrased as asserting that, in a complete metric space, a countable intersection of dense $G_\delta$’s is still a dense $G_\delta$.

[12.1] Theorem: (Baire) Let $X$ be either a complete metric space or a locally compact Hausdorff topological space. The intersection of a countable collection $U_1, U_2, \ldots$ of dense open subsets $U_i$ of $X$ is still dense in $X$.

Proof: Let $B_o$ be a non-empty open set in $X$, and show that $\bigcap_i U_i$ meets $B_o$. Suppose that we have inductively chosen an open ball $B_{n-1}$. By the denseness of $U_n$, there is an open ball $B_n$ whose closure $\overline{B_n}$ satisfies

$$\overline{B_n} \subset B_{n-1} \cap U_n$$

Further, for complete metric spaces, take $B_n$ to have radius less than $1/n$ (or any other sequence of reals going to 0), and in the locally compact Hausdorff case take $B_n$ to have compact closure.

Let

$$K = \bigcap_{n \geq 1} \overline{B_n} \subset B_o \cap \bigcap_{n \geq 1} U_n$$

For complete metric spaces, the centers of the nested balls $B_n$ form a Cauchy sequence (since they are nested and the radii go to 0). By completeness, this Cauchy sequence converges, and the limit point lies inside each closure $\overline{B_n}$, so lies in the intersection. In particular, $K$ is non-empty. For locally compact Hausdorff spaces, the intersection of a nested family of non-empty compact sets is non-empty, so $K$ is non-empty, and $B_o$ necessarily meets the intersection of the $U_n$. \///

13. Urysohn’s lemma

Urysohn’s lemma proves existence of sufficiently many functions on reasonable topological spaces.

[13.1] Theorem: (Urysohn) In a locally compact Hausdorff topological space $X$, given a compact subset $K$ contained in an open set $U$, there is a continuous function $0 \leq f \leq 1$ which is 1 on $K$ and 0 off $U$.

Proof: First, we prove that there is an open set $V$ such that

$$K \subset V \subset \overline{V} \subset U$$
For each \( x \in K \) let \( V_x \) be an open neighborhood of \( x \) with compact closure. By compactness of \( K \), some finite subcollection \( V_{x_1}, \ldots, V_{x_n} \) of these \( V_x \) cover \( K \), so \( K \) is contained in the open set \( W = \bigcup_{i} V_{x_i} \) which has compact closure \( \bigcup_i V_{x_i} \), since the union is finite.

Using the compactness again in a similar fashion, for each \( x \) in the closed set \( X - U \) there is an open \( W_x \) containing \( K \) and a neighborhood \( U_x \) of \( x \) such that \( W_x \cap U_x = \phi \).

Then
\[
\bigcap_{x \in X - U} (X - U) \cap W \cap W_x = \phi
\]

These are compact subsets in a Hausdorff space, so (again from compactness) some finite subcollection has empty intersection, say
\[
(X - U) \cap (W \cap W_{x_1} \cap \ldots \cap W_{x_n}) = \phi
\]
That is,
\[
W \cap W_{x_1} \cap \ldots \cap W_{x_n} \subset U
\]
Thus, the open set
\[
V = W \cap W_{x_1} \cap \ldots \cap W_{x_n}
\]
meets the requirements.

Using the possibility of inserting an open subset and its closure between any \( K \subset U \) with \( K \) compact and \( U \) open, we inductively create opens \( V_r \) (with compact closures) indexed by rational numbers \( r \) in the interval \( 0 \leq r \leq 1 \) such that, for \( r > s \),
\[
K \subset V_r \subset V_r \subset V_s \subset V_s \subset U
\]
From any such configuration of opens we construct the desired continuous function \( f \) by
\[
f(x) = \sup\{r \text{ rational in } [0,1] : x \in V_r, \} = \inf\{r \text{ rational in } [0,1] : x \in V_r, \}
\]
It is not immediate that this sup and inf are the same, but if we grant their equality then we can prove the continuity of this function \( f(x) \). Indeed, the sup description expresses \( f \) as the supremum of characteristic functions of open sets, so \( f \) is at least lower semi-continuous. \[8\] The inf description expresses \( f \) as an infimum of characteristic functions of closed sets so is upper semi-continuous. Thus, \( f \) would be continuous.

To finish the argument, we must construct the sets \( V_r \) and prove equality of the inf and sup descriptions of the function \( f \).

To construct the sets \( V_i \), start by finding \( V_0 \) and \( V_1 \) such that
\[
K \subset V_1 \subset V_1 \subset V_0 \subset V_0 \subset U
\]
Fix a well-ordering \( r_1, r_2, \ldots \) of the rationals in the open interval \( (0,1) \). Supposing that \( V_{r_1}, \ldots, V_{r_n} \) have been chosen, let \( i, j \) be indices in the range \( 1, \ldots, n \) such that
\[
r_j > r_{n+1} > r_i
\]
and \( r_j \) is the smallest among \( r_1, \ldots, r_n \) above \( r_{n+1} \), while \( r_i \) is the largest among \( r_1, \ldots, r_n \) below \( r_{n+1} \). Using the first observation of this argument, find \( V_{r_{n+1}} \) such that
\[
V_{r_j} \subset V_{r_j} \subset V_{r_{n+1}} \subset V_{r_{n+1}} \subset V_{r_i} \subset V_{r_i}
\]
\[8\] A (real-valued) function \( f \) is lower semi-continuous when for all bounds \( B \) the set \( \{x : f(x) > B\} \) is open. The function \( f \) is upper semi-continuous when for all bounds \( B \) the set \( \{x : f(x) < B\} \) is open. It is easy to show that a sup of lower semi-continuous functions is lower semi-continuous, and an inf of upper semi-continuous functions is upper semi-continuous. As expected, a function both upper and lower semi-continuous is continuous.
This constructs the nested family of opens.

Let \( f(x) \) be the sup and \( g(x) \) the inf of the characteristic functions above. If \( f(x) > g(x) \) then there are \( r > s \) such that \( x \in V_r \) and \( x \notin \overline{V_s} \). But \( r > s \) implies that \( V_r \subset \overline{V_s} \), so this cannot happen. If \( g(x) > f(x) \), then there are rationals \( r > s \) such that

\[
g(x) > r > s > f(x)
\]

Then \( s > f(x) \) implies that \( x \notin V_s \), and \( r < g(x) \) implies \( x \in \overline{V_r} \). But \( r > s \) implies that \( V_r \subset \overline{V_s} \), so this cannot happen. If \( g(x) > f(x) \), then there are rationals \( r > s \) such that \( g(x) > r > s > f(x) \). Then \( s > f(x) \) implies that \( x \notin V_s \), and \( r < g(x) \) implies \( x \in \overline{V_r} \). But \( V_r \subset \overline{V_s} \), contradiction. Thus, \( f(x) = g(x) \). ///

14. Appendix: mapping-property characterization of completion

Our intention is that, when a metric space \( X \) is not complete, there should be a complete metric space \( \overline{X} \) and an isometry (distance-preserving) \( j : X \to \overline{X} \), such that every isometry \( f : X \to Y \) to complete metric space \( Y \) factors through \( j \) uniquely. That is, there are commutative diagrams\[^9\] of continuous maps

\[
\begin{array}{ccc}
\overline{X} & \xrightarrow{\exists!} & Y \\
| & \downarrow{\exists!} & \\
X & \xrightarrow{\forall} & Y \\
\end{array}
\quad \text{for every isometry } X \to Y
\]

Without describing any constructions of completions, we can prove some things about the behavior of any possible completion. In particular, we prove that any two completions are naturally isometrically isomorphic to each other. Thus, the outcome will be independent of construction.

\[14.1\] Claim: (Uniqueness) Let \( i : X \to Y \) and \( j : X \to Z \) be two completions of a metric space \( X \). Then there is a unique isometric homeomorphism \( h : Y \to Z \) such that \( j = h \circ i \). That is, we have a commutative diagram

\[
\begin{array}{ccc}
Y & \xrightarrow{\exists!} & Z \\
| & \downarrow{\exists!} & \downarrow{f} \\
X & \xrightarrow{i} & Y \\
\end{array}
\]

Proof: First, take \( Y = Z \) and \( f : X \to Y \) to be the inclusion \( i \), in the characterization of \( i : X \to Y \). The characterization of \( i : X \to Y \) shows that there is unique isometry \( f : Y \to Y \) fitting into a commutative diagram

\[
\begin{array}{ccc}
Y & \xrightarrow{\exists!} & Y \\
| & \downarrow{f} & \\
X & \xrightarrow{i} & Y \\
\end{array}
\]

Since the identity map \( Y \to Y \) certainly fits into this diagram, the only map \( f \) fitting into the diagram is the identity on \( Y \).

\[^9\] A diagram of maps is commutative when the composite map from one object to another within the diagram does not depend on the route taken within the diagram.
Next, applying the characterizations of both $i : X \to Y$ and $j : X \to Z$, we have unique $f : Y \to Z$ and $g : Z \to Y$ fitting into

```
  Y
  i
  j
  X  f
  \downarrow
  \downarrow
  Z
```

Then $f \circ g : Y \to Y$ and $g \circ f : Z \to Z$ fit into

```
  Y
  i
  j
  X  f
  \downarrow
  \downarrow
  Z
```

By the first observation, this means that $f \circ g$ is the identity on $Y$, and $g \circ f$ is the identity on $Z$, so $f$ and $g$ are mutual inverses, and $Y$ and $Z$ are homeomorphic.

[14.2] Remark: A virtue of the characterization of completion is that it does not refer to the *internals* of any completion.

Next, we see that the mapping-property characterization of a completion does not introduce more points than absolutely necessary:

[14.3] Claim: Every point in a completion $\tilde{X}$ of $X$ is the limit of a Cauchy sequence in $X$. That is, $X$ is dense in $\tilde{X}$.

Proof: Write $d(,)$ for both the metric on $X$ and its extension to $\tilde{X}$. Let $Y \subset \tilde{X}$ be the collection of limits of Cauchy sequences of points in $X$. We claim that $Y$ itself is complete. Indeed, given a Cauchy sequence \{yi\} in $Y$ with limit $z \in \tilde{X}$, let $x_i \in X$ such that $d(x_i, y_i) < 2^{-i}$. It will suffice to show that \{xi\} is Cauchy with limit $z$. Indeed, given $\varepsilon > 0$, take $N$ large enough so that $d(y_i, z) < \varepsilon/2$ for all $i \geq N$, and increase $N$ if necessary so that $2^{-i} < \varepsilon/2$. Then, by the triangle inequality, $d(x_i, z) < \varepsilon$ for all $i \geq N$. Thus, $Y$ is complete.

By the defining property of $\tilde{X}$, every isometry $f : X \to Z$ to complete $Z$ has a unique extension to an isometry $F : \tilde{X} \to Z$ fitting into

```
  Y
  i
  j
  X  f
  \downarrow
  \downarrow
  \tilde{X}
```

Since $Y$ is already complete and $j(X) \subset Y$, the restriction of $F$ to $Y$ gives a diagram

```
  Y
  j
  X  f
  \downarrow
  \downarrow
  Z
```

That is, $Y$ fits the characterization of a completion of $X$. By uniqueness, $Y \subset \tilde{X}$ is a homeomorphism, so $Y = \tilde{X}$.

[15. Appendix: why the product topology is so coarse](#)
Often it is not the *internal structure* of a thing that is interesting, but its *interactions* with other objects. That is, often we have little long-term interest in the details of a *construction* of the thing, but care more about how it *behaves*. Thus, to express our genuine intentions, we should *not* first *construct* the thing, and only gradually admit that it does what we had planned all along. Instead, we should tell what *external interactions* we demand or expect, and worry about internal details later.

Happily, often the characterizations of an object in terms of maps to and from other objects of the same sort succeed in uniquely determining the thing. Even more surprisingly, often this uniqueness follows merely from the shape of the diagrams of the maps, not from any subtler or internal features of the maps or objects.

For example, the nature of the *product topology* on products of topological spaces is illuminated by this approach. In particular, one might have (at some point) wondered *why the product topology is so coarse*. That is, on infinite products the product topology is strictly coarser than the *box topology*. The answer is that the product topology is *what it has to be*. That is, there is no genuine *choice* in the construction. Of course, this needs explanation.

### [15.1] Definition and uniqueness of products

A *product* of non-empty topological spaces $X_\alpha$ for $\alpha$ in an index set $A$ is a topological space $X$ with (*projection*) maps $p_\alpha : X \rightarrow X_\alpha$, such that every family $f_\alpha : Z \rightarrow X_\alpha$ of maps from some other topological space *factors through* the $p_\alpha$ *uniquely*, in the sense that there is a unique $f : Z \rightarrow X$ such that

$$f_\alpha = p_\alpha \circ f \quad \text{(for all $\alpha$)}$$

Pictorially, one says that *all triangles commute* in the diagram\[10\]

![Diagram](image)

### [15.2] Claim: There is at most one product of given spaces $X_\alpha$, up to *unique* isomorphism.

That is, given two products, $X$ with projections $p_\alpha$ and $Y$ with projections $q_\alpha$, there is a unique isomorphism $i : X \rightarrow Y$ respecting the projections in the sense that

$$p_\alpha = q_\alpha \circ i \quad \text{(for all $\alpha$)}$$

That is, there is a unique isomorphism $i : X \rightarrow Y$ such that all triangles commute in the diagram

![Diagram](image)

\[10\] Often an arrow is drawn with a dotted line rather than a solid line to suggest that it is a consequence of or is derived from the other maps in the diagram.
Proof: First, we prove that products have no (proper) endomorphisms, meaning that the identity map is the only map \( \varphi : X \to X \) making all triangles commute in the diagram

Indeed, using \( X \) and the \( p_\alpha \) in the role of \( Z \) and \( f_\alpha \) in the defining property of the product, we have a unique \( \varphi \) making all triangles commute in

The identity map certainly meets the requirement on \( \varphi \), so, by uniqueness, the identity map on \( X \) is the only such map \( \varphi \).

Next, let \( Y \) be another product, with projections \( q_\alpha \). Letting \( Y \) take the role of \( Z \) and \( q_\alpha \) the role of \( f_\alpha \) in the defining property of the product \( X \) (with its \( p_\alpha \)'s), we have a unique \( q : Y \to X \) such that all triangles commute in

Reversing the roles of \( X \) and \( Y \) (and their projections), we also have a unique \( p : X \to Y \) such that (this time in symbols rather than the picture) \( p_\alpha = q_\alpha \circ p \) for all \( \alpha \).

Then \( p \circ q : Y \to Y \) is an endomorphism of \( Y \) respecting the projections \( q_\alpha \), since

Thus, \( p \circ q \) must be the identity on \( Y \). Similarly, \( q \circ p \) is the identity on \( X \). Thus, \( p \) and \( q \) are mutual inverses, so both are isomorphisms. And we did have uniqueness, along the way. ///

[15.3] Remark: We did not use any features of topological spaces nor of continuous maps in this proof. Instead, the quantification over all \( Z \) and maps \( f_\alpha \) indirectly described what was happening. Thus, we discover that a similar result holds for any prescribed collection of things with prescribed maps between

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Note that the uniqueness proof just given applies immediately to sets without topologies, verify properties, rather merely set isomorphism.

[15.4] Remark: Since there is at most one product, the issue of figuring out what it is will be simpler than if there were many possibilities. Further, we can use the mapping properties to find hints about a construction, thus proving existence.

[15.5] Construction of products of sets

Before addressing the topology on the product, we first construct it as a set. Of course, we secretly know that it is the usual Cartesian product, but it is interesting to see that we can discover this from the mapping properties, rather merely verify that the Cartesian product fits (which we do at the end).

Note that the uniqueness proof just given applies immediately to sets without topologies, as well. That is, the same diagrams but with objects just sets and maps arbitrary set maps, define a product $X$ (with projections $p_\alpha$) of non-empty sets $X_\alpha$, and the same argument proves that there is at most one such thing (up to unique isomorphism). And, yes, we secretly know that the product is the usual Cartesian product, and the projections are the obvious things, but we also want to see how one might be led to this understanding.

To investigate properties of a product $X$ of non-empty sets $X_\alpha$ (with projections $p_\alpha$), we should consider various sets $Z$ and maps $f_\alpha : Z \to X_\alpha$ to see what we learn about $X$ as by-products. In this austere setting, there are not very many choices that suggest themselves, but, on the other hand, there are not many wrong choices available, either.

For example, given $x \in X$, we have the collection of all projections’ values $p_\alpha(x)$, with unclear relations (or none at all) among these values. For example, to see whether things get mashed down, and as an exercise in technique, we could try to prove

[15.6] Claim: For $x \neq y$ both in $X$, there is at least one $\alpha \in A$ such that $p_\alpha(x) \neq p_\alpha(y)$.

Proof: Suppose that $p_\alpha(x) = p_\alpha(y)$ for all $\alpha \in A$. Let $S = \{s\}$ be a set with one element, and define $f_\alpha : S \to X_\alpha$ by

$$f_\alpha(s) = p_\alpha(x) \quad (= p_\alpha(y), \text{ also})$$

Then, by definition, there is a unique $f : S \to X$ such that $f_\alpha = p_\alpha \circ f$ for all $\alpha$. But notice that $f$ defined by $f(s) = x$ and also $f$ defined by $f(s) = y$ have this property. By uniqueness of $f$, we have $x = f(s) = y$. ///

From the other side, we can wonder whether all possible collections of values of projections can occur. Intuitively (and secretly knowing the answer in advance) we might doubt that there are any constraints, but we are obliged to demonstrate this by exhibiting maps.

[15.7] Claim: Given choices $x_\alpha \in X_\alpha$ for all $\alpha \in A$, there is $x$ in the product such that $p_\alpha(x) = x_\alpha$ for all $\alpha$. (And by the previous claim this $x$ is unique.)

Proof: Again, let $S = \{s\}$ be a set with a single element, and define $f_\alpha : S \to X_\alpha$ by $f_\alpha(s) = x_\alpha$. Then there exists a unique $f : S \to X$ such that $f_\alpha = p_\alpha \circ f$. That is

$$x_\alpha = f_\alpha(s) = (p_\alpha \circ f)(s) = p_\alpha(f(s))$$

The element $x = f(s)$ is the desired one. ///

In fact, these two claims, with their proofs, suggest that the product $X$ is exactly the collection of all choices of families $\{f_\alpha : \alpha \in A\}$ of functions $f_\alpha : \{s\} \to X_\alpha$, and $\alpha^{th}$ projection given by

$$p_\alpha : \{f_\alpha : \alpha \in A\} \to f_\alpha(s)$$

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This is correct, and by uniqueness we know that any other construction must give an isomorphic thing, but this can be simplified, since maps from a one-element set are entirely determined by their images. Thus, finally, we have been led back to the Cartesian product

\[ X = \{ \{ x_\beta : \beta \in A \} : x_\beta \in X_\beta \} \]

and usual projections

\[ p_\alpha(\{ x_\beta : \beta \in A \}) = x_\alpha \]

And, finally, we give the trivial proof of

[15.8] Claim: The Cartesian product and usual projections are a product for sets.

Proof: Given a family of maps \( f_\alpha : Z \to X_\alpha \), define \( f : Z \to X \) by

\[ f(z) = \{ f_\beta(z) : \beta \in A \} \in X \]

This meets the defining condition, and is visibly the only map that will do so. ///

[15.9] Remark: The point here was that there is no alternative, up to (unique!) isomorphism, and that some reasonable considerations based on the mapping-property definition can lead us to a construction.

[15.10] Remark: The little fact that the collection of set maps \( \varphi \) from \( S = \{ s \} \) to a given set \( Y \) is isomorphic to \( Y \) (via \( \varphi \to \varphi(s) \)) is noteworthy in itself.

[15.11] Construction of product topologies

So, at last, what topology does the mapping-property definition require on the Cartesian product \( X \) of the underlying sets for non-empty topological spaces \( X_\alpha \)?

One should be aware that it is possible to apparently define non-existent (impossible) objects by mapping properties, and the impossibility may not be immediately clear. Thus, one should have respect for the problem of constructing objects whose uniqueness (if they exist) is much easier.

First, all the projections \( p_\alpha : X \to X_\alpha \) defined as expected by

\[ p_\alpha(\{ x_\beta : \beta \in A \}) = x_\alpha \]

must be continuous. That is, for every open set \( U_\alpha \) in \( X_\alpha \), \( p_\alpha^{-1}(U_\alpha) \) is open in \( X \). Note that

\[ p_\alpha^{-1}(U_\alpha) = \prod_{\beta \in A} U_\beta \quad \text{(where } U_\beta = X_\beta \text{ for } \beta \neq \alpha) \]

Thus, the topology on \( X \) must contain at least these sets as opens, which does entail that the topology include finite intersections of them, which are exactly sets of the form

\[ \prod_{\beta \in A} U_\beta \quad \text{(with } U_\beta \text{ open in } X_\beta, \text{ and } U_\beta = X_\beta \text{ for all but finitely-many } \beta) \]

And arbitrary unions of these finite intersections must be included in the topology, and so on. Thus, the product topology is at least as fine as the topology generated by the sets \( p_\alpha^{-1}(U_\alpha) \).

From the other side, the condition concerning maps from another space \( Z \) gives a constraint on how coarse the topology must be, as follows. Given a family of continuous maps \( f_\alpha : Z \to X_\alpha \), the corresponding \( f : Z \to X \) must be continuous, which requires by definition that \( f^{-1}(U) \) must be open in \( Z \) for all opens \( U \) in \( X \). If there were too many opens \( U \) in \( X \) this condition could not be met. So the topology on \( X \) must
be at least as coarse as would be allowed by \( f^{-1}(U) \) being open in \( Z \) for all \( f : Z \to X \). Yes, this is a less tangible constraint, since it is hard to visualize the quantification over all \( f : Z \to X \).

It is reasonable to hope that the explicit topology on \( X \) generated by the inverse images \( p^{-1}_\alpha(U) \) can be proven to be sufficiently coarse to meet the second condition.

To prove continuity of \( f : Z \to X \), thus proving that the topology on \( X \) is sufficiently coarse, it suffices to prove that \( f^{-1}(U) \) is open for all opens \( U \) in a sub-basis. And, happily, by \( p_\alpha \circ f = f_\alpha \),

\[
f^{-1}(p^{-1}_\alpha(U_\alpha)) = f^{-1}_\alpha(U_\alpha)
\]

which is open by the continuity of \( f_\alpha \). So \( f \) is continuous, which is to say that the topology on \( X \) is coarse enough (not too fine), so succeeds in being suitable for a product. ///

[15.12] Remark: Thus, the product topology must be fine enough so that inverse images \( p_\alpha(U_\alpha) \) in \( X \) of projections \( p_\alpha : X \to X_\alpha \) are open, and coarse enough so that inverse images \( f^{-1}(U) \) in \( Z \) of induced maps \( f : Z \to X \) are open.

[15.13] Remark: The main issue here is existence of any topology at all that will work as a product. By contrast, the uniqueness of products (if they exist) was proven earlier, in a standard (even clichéd) mapping-property fashion. That is, the uniqueness up to unique isomorphism asserts that if \( X \) and \( Y \) were two products, they must be (uniquely) homeomorphic.

[15.14] Remark: To repeat: the question of existence of a product topology can be viewed as being the question of whether or not the topology generated by the sets \( p^{-1}_\alpha(U_\alpha) \) might accidentally be too fine for all induced maps \( Z \to X \) to be continuous. That is, the continuity of the projections and the continuity of the induced maps \( Z \to X \) are opposing constraints. The for-general-reasons uniqueness tells us a priori that there is at most one simultaneous solution, so the question is whether or not there is any. In this case, it turns out that these conflicting constraints do allow a common solution. Still, it does happen in other circumstances that two opposing conditions allow no simultaneous satisfaction.

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16. Recap: some important function spaces and metrics

All functions here are real-valued or complex-valued.

For a topological space \( X \), \( C^\infty(X) \) is the space of continuous functions on \( X \). Some sources omit the superscript. For compact \( X \), one natural norm on \( C^\infty(X) \) is the sup-norm

\[
|f|_\infty = |f|_{sup} = |f|_{C^\infty} = \sup_{x \in X} |f(x)|
\]

and we have seen that \( C^\infty(X) \) is complete for the corresponding metric.

For not-necessarily-compact \( X \), \( C^0(X) \) is the space of continuous bounded functions on \( X \), with sup-norm.

For not-necessarily-compact \( X \), \( C^0_{bld}(X) \) is the space of continuous functions on \( X \) with compact support, with sup-norm. As usual, the support of a function is the closure of the subset of \( X \) on which the function is non-zero.

For not-necessarily-compact \( X \), \( C^0(X) \) is the space of continuous functions on \( X \) going to zero at infinity, with sup-norm. Going to zero at infinity means that, given \( \varepsilon > 0 \), there is a compact \( K \subset X \) such that, for all \( x \notin K \), \(|f(x)| < \varepsilon\).

For \( k \in \{0, 1, 2, \ldots\} \), and for \( X \) admitting a differentiable structure, as do intervals \([a, b]\), spaces \( \mathbb{R}^n \), and so on, \( C^k(X) \) is the space of \( k \)-times continuously differentiable functions, and \( C^\infty(X) \) is the space of smooth functions, that is, infinitely or indefinitely continuously differentiable.
For $X = [a, b]$, $C^k(X)$ has a good natural norm

$$|f|_{C^k} = |f|_{C^0} + |f'|_{C^0} + |f''|_{C^0} + \ldots + |f^{(k)}|_{C^0}$$

It takes a bit of work, but it is complete.

The space $C_c^\infty(\mathbb{R}^n)$ of test functions on $\mathbb{R}^n$ is the space of compactly supported smooth functions.

The $L^2$-metric, for example on $C^0[a, b]$, is

$$|f|_{L^2} = \sqrt{\int_a^b |f(x)|^2_C\, dx}$$

More generally, for $1 \leq p < +\infty$, the $L^p$-metric on $C^0[a, b]$ is

$$|f|_{L^p} = \sqrt[p]{\int_a^b |f(x)|^2_C\, dx}$$

In a different vein,

$$\ell^2 = \{ (c_1, c_2, \ldots) : \text{with } c_i \in \mathbb{C} \text{ and } \sum_n |c_n|^2 < +\infty \}$$

and the norm is $|(c_1, \ldots)|_{\ell^2} = \sqrt{|c_0|^2}$. Similarly, for $1 \leq p < \infty$, we obtain the spaces $\ell^p$ by replacing squares and square roots by $p$ and $p^{th}$ roots. As a limiting case,

$$\ell^\infty = \{ (c_1, c_2, \ldots) : \text{with } c_i \in \mathbb{C} \text{ and } \sup_n |c_n| < +\infty \}$$

with norm $|(c_1, \ldots)|_{\ell^\infty} = \sup_n |c_n|$.

[16.1] Remark: This is not an exhaustive list!

[16.2] Remark: In some cases, such as the $L^p$ spaces, in the present context we put the metric on continuous functions only because we know very well how to integrate them.