03b. Completeness of $L^p$ spaces

Paul Garrett  garrett@math.umn.edu  http://www.math.umn.edu/~garrett/

[This document is http://www.math.umn.edu/~garrett/m/real/notes_2019-20/03b_completeness_of_Lp.pdf]

1. Examples: spaces $L^p$

Given a measure space $X$, for $1 \leq p < \infty$ the usual $L^p$ spaces are

$$L^p(X) = \{\text{measurable } f : |f|_{L^p} < \infty\} \text{ modulo } \sim$$

with the usual $L^p$ norm

$$|f|_{L^p} = \left(\int_X |f|^p\right)^{1/p}$$

and associated metric

$$d(f, g) = |f - g|_{L^p}$$

taking the quotient by the equivalence relation

$$f \sim g \text{ if } f - g = 0 \text{ off a set of measure 0}$$

[1.1] Remark: For general measure spaces this is not a metric until we take the quotient, since, otherwise, two different functions differing only on a set of measure 0 would be distance 0 from each other, but would not be equal.

[1.2] Remark: These $L^p$ functions have inevitably ambiguous pointwise values, in conflict with the naive formal definition of function. Nevertheless, one usually does think of $L^p$ functions as being more-or-less functions.

A simple instance of this construction, for a measure that has no sets of measure 0, so needs no quotient, is

$$\ell^p = \{\text{complex sequences } \{c_i\} \text{ with } \sum_i |c_i|^p < \infty\}$$

with norm $|(c_1, c_2, \ldots)|_{\ell^p} = \left(\sum_i |c_i|^p\right)^{1/p}$. The analogue of the following theorem for $\ell^p$ is more elementary.

[1.3] Theorem: The space $L^p(X)$ is a complete metric space.

[1.4] Remark: In fact, as used in the proof, a Cauchy sequence $f_i$ in $L^p(X)$ has a subsequence converging pointwise off a set of measure 0 in $X$.

Proof: The triangle inequality here is Minkowski’s inequality. To prove completeness, choose a subsequence $f_{n_i}$ such that

$$|f_{n_i} - f_{n_{i+1}}|_p < 2^{-i}$$

and put

$$g_n(x) = \sum_{1 \leq i \leq n} |f_{n_{i+1}}(x) - f_{n_i}(x)|$$

and

$$g(x) = \sum_{1 \leq i < \infty} |f_{n_{i+1}}(x) - f_{n_i}(x)|$$
The infinite sum is not necessarily claimed to converge to a finite value for every \( x \). The triangle inequality shows that \( |g_n|_p \leq 1 \). Fatou’s Lemma asserts that for \([0, \infty]\)-valued measurable functions \( h_i \)

\[
\int_X \left( \lim \inf_i h_i \right) \leq \lim \inf_i \int_X h_i
\]

Thus, \( |g|_p \leq 1 \), so is finite. Thus,

\[
f_{n_1}(x) + \sum_{i \geq 1} (f_{n_{i+1}}(x) - f_{n_i}(x))
\]

converges for almost all \( x \in X \). Let \( f(x) \) be the sum at points \( x \) where the series converges, and on the measure-zero set where the series does not converge put \( f(x) = 0 \). Certainly

\[
f(x) = \lim_i f_{n_i}(x) \quad \text{(for almost all } x)\]

Now prove that this almost-everywhere pointwise limit is the \( L^p \)-limit of the original sequence. For \( \varepsilon > 0 \) take \( N \) such that \( |f_m - f_n|_p < \varepsilon \) for \( m, n \geq N \). Fatou’s lemma gives

\[
\int |f - f_n|^p \leq \lim \inf_i \int |f_{n_i} - f_n|^p \leq \varepsilon^p
\]

Thus \( f - f_n \) is in \( L^p \) and hence \( f \) is in \( L^p \). And \( |f - f_n|_p \to 0 \). ///

[1.5] **Theorem:** For a locally compact Hausdorff topological space \( X \) with positive regular Borel measure \( \mu \), the space \( C^0_c(X) \) of compactly-supported continuous functions is dense in \( L^1(X, \mu) \).

**Proof:** From the definition of integral attached to a measure, an \( L^1 \) function is approximable in the \( L^1 \)-metric by a simple function, that is, a measurable function assuming only finitely-many values. That is, a simple function is a finite linear combination of characteristic functions of measurable sets \( E \). Thus, it suffices to approximate characteristic functions of measurable sets by continuous functions. The assumed regularity of the measure gives compact \( K \) and open \( U \) such that \( K \subset E \subset U \) and \( \mu(U - E) < \varepsilon \), for given \( \varepsilon > 0 \). Urysohn’s lemma says that there is continuous \( f \) identically 1 on \( K \) and identically 0 off \( U \). Thus, \( f \) approximates the characteristic function \( \chi_E \) of \( E \) in \( L^1 \):

\[
|f - \chi_E|_L^1 = \int_X |f - \chi_E| = \int_{U - K} |f - \chi_E| = \int_{U - K} 1 < \varepsilon
\]

///

[1.6] **Corollary:** For locally compact Hausdorff \( X \) with regular Borel measure \( \mu \), \( L^1(X, \mu) \) is the \( L^1 \)-metric completion of \( C^0_c(X) \), the compactly-supported continuous functions. ///

[1.7] **Remark:** Defining \( L^1(X, \mu) \) to be the \( L^p \) completion of \( C^0_c(X) \) avoids discussion of ambiguous values on sets of measure zero, but also leaves ambiguity about in what sense the completion consists of functions.