08f. Poisson summation by distribution theory

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Let \( \psi(y) = e^{2\pi iy} \) and \( \psi_x(y) = \psi(xy) \). The additive group \( \mathbb{R} \) acts on \( \mathcal{S} \) and on \( \mathcal{D} \) continuously by the regular representation \( R_g f(x) = f(x + g) \). The natural duality gives the (continuous) dual or contragredient representation on \( \mathcal{S}^* \) and \( \mathcal{D}^* \) by
\[
(R^*_g u)(f) = u(R_{-g} f)
\]

There are two fundamental identities regarding this regular representation and Fourier transforms (for \( f \in \mathcal{S} \)):
\[
(R_x f)^\wedge = \psi_x \hat{f} \\
(\psi_x f) = R_{-x} \hat{f}
\]

From these and from the definition of Fourier transform for tempered distributions, the same identities hold for tempered distributions, as well.

For a collection \( \Phi \) of smooth functions on \( \mathbb{R} \) with common zero set \( Z \), and a distribution \( u \) such that \( \varphi u = 0 \) for all \( \varphi \in \Phi \), \( \text{spt} u \subset Z \). In particular, for \( \Phi \) a subset of \( C^\infty_0(\mathbb{R}) \) having a single point \( \{0\} \) as common zero set. Let \( O_0 \) be the ring of germs of smooth functions at 0, and let \( \mathfrak{m} \) be its unique maximal ideal, consisting of smooth functions vanishing at 0. Suppose that the image in \( O_0 \) of the ideal generated by \( \Phi \) in \( C^\infty_0(\mathbb{R}) \) is exactly \( \mathfrak{m}^n \). That is, we suppose that all functions in \( \Phi \) vanish at 0 to order at least \( n \), and every germ of a smooth function at 0 vanishing to order at least \( n \) is a linear combination over \( O_0 \) of elements of \( \Phi \). Let \( u \) be a distribution so that \( \varphi u = 0 \) for all \( \varphi \in \Phi \). Then \( u \) is a complex linear combination of \( \delta_0, \ldots, \delta^{(n-1)}_0 \).

Consider the tempered distributions
\[
u(f) = \sum_{n \in \mathbb{Z}} f(n) \quad \text{and} \quad v(f) = \sum_{n \in \mathbb{Z}} \hat{f}(n)
\]
The Poisson summation formula asserts that \( u = v \). We will identify properties possessed by both \( u \) and \( v \), and prove that there is a unique tempered distribution with these properties.

Certainly \( u(\psi_n f) = u(f) \) for \( n \in \mathbb{Z} \), and \( u(R_n f) = u(f) \) for \( n \in \mathbb{Z} \). Thus,
\[
\psi_n u = u \quad R_n u = u \quad \text{for all} \ n \in \mathbb{Z}
\]
The two identities above which intertwine Fourier transform and the regular representation imply that \( v \) has the same properties. Further, letting \( \gamma(x) := e^{-\pi x^2} \), we have \( \hat{\gamma} = \gamma \), and so \( u(\gamma) = v(\gamma) \).

Now we prove that the space of tempered distributions \( w \) such that
\[
\psi_n w = w \quad R_n w = w \quad \text{for all} \ n \in \mathbb{Z}
\]
is one-dimensional over \( \mathbb{C} \). This, together with the evaluation of both \( u \) and \( v \) on \( \gamma \), will prove the Poisson summation formula.

The common zero set of the collection \( \Phi = \{ \psi_n - 1 : n \in \mathbb{Z} \} \) is \( \mathbb{Z} \), so any distribution \( w \) annihilated by multiplication by all \( \psi_n - 1 \) must be supported on \( \mathbb{Z} \). Let \( \varphi \in C^\infty_0(\mathbb{R}) \) be such that \( \text{spt} \varphi \cap \mathbb{Z} = \{0\} \) and \( \varphi = 1 \) on some neighborhood of 0. Then \( \text{spt}(\varphi w) = \{0\} \). Further, since the \( \psi_n - 1 \) generate the whole maximal ideal in the ring of germs of smooth functions at 0, we conclude that \( \varphi w \) is a constant multiple of \( \delta \). By use of a partition of unity to localize the issues, we find that
\[
w = \sum_c c_n \delta_n
\]
for some constants $c_n$. The translation invariance of $w$ implies that all the constants $c_n$ are the same. Thus, there is a constant $c$ so that

$$w = c \sum \delta_n$$

This is the desired uniqueness (one-dimensionality) assertion.