09c. The family \( \sin \frac{nx}{2} \)

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1. Boundary value problem \( u'' - \lambda u = f \), \( u(0) = 0 = u(2\pi) \)
2. The Green's function

Essentially the only general device to prove that a collection \( \{u_n\} \) of functions on \([a,b]\) is an orthogonal basis for \( L^2[a,b] \) is to find a \textit{self-adjoint compact} operator \( T \) on \( L^2[a,b] \) such that the eigenvectors of \( T \) are the functions \( u_n \).

The explicit Fourier-Dirichlet and Fejer arguments for ordinary Fourier series do not easily generalize. One family of natural extensions is to Sturm-Liouville boundary-value problems (1830s), but there already we see the necessity of the spectral theory of \textit{self-adjoint compact operators} on Hilbert spaces, even though such ideas and proofs had to wait almost 60 years (1890s), for Bocher and Steklov, and then Hilbert and Schmidt and others a few years later.

As a non-trivial but relatively simple example, we prove that the functions \( u_n(x) = \sin \frac{n\pi x}{2} \) give an orthogonal basis of \( L^2[0,2\pi] \). This does \textit{not} follow in any easy way from the corresponding fact for \( \{e^{inx}\} \) or for \( \sin nx \) and \( \cos nx \), despite some mythology to the contrary.

1. Boundary value problem \( u'' - \lambda u = f \), \( u(0) = 0 = u(2\pi) \)

Consider the boundary-value problem \( u'' - \lambda u = f \) with \( u(0) = u(2\pi) = 0 \) on \([0,2\pi]\), and \( \lambda \in \mathbb{C} \). Naively formulated, given a function \( f \) on \([0,2\pi]\), this asks to solve \( u'' - \lambda u = f \) subject to the boundary conditions \( u(0) = u(2\pi) = 0 \). This formulation is naive because we have not specified what kind of \textit{functions} \( f \) and \( u \) should be, apart from the apparent requirement that requiring vanishing of \( u \) at endpoints has sense to it, and that \( u \) admits derivatives of some sort.

The eigenvectors for \( u \to u'' \) are not hard to find: the differential equation \( u'' - \lambda u = f \) has linearly independent solutions \( u(x) = e^{\pm \sqrt{\lambda}x} \) for \( \lambda \neq 0 \), and \( u(x) = ax + b \) for \( \lambda = 0 \). In fact, these are also the only \textit{distributional} eigenvectors.

The Dirichlet boundary conditions (that is, vanishing at endpoints) exclude the \( \lambda = 0 \) eigenvectors. For \( \lambda = s^2 \neq 0 \), the requirement is to find constants \( A, B \) (not both 0) such that

\[
Ae^{s0} + Be^{-s0} = 0 = Ae^{s2\pi} + Be^{-s2\pi}
\]

Thus, \( B = -A \), and \( e^{s2\pi} - e^{-s2\pi} = 0 \). Equivalently, \( e^{2s\pi} = 1 \). Thus, \( s \in \mathbb{Z} \). That is, \( u_n(x) = \sin \frac{n\pi x}{2} \) for \( n = 1, 2, 3, \ldots \) are all the eigenvectors of \( u \to u'' \) meeting the Dirichlet boundary conditions, and have eigenvalues \( n^2/4 \).

These eigenvectors are mutually orthogonal, either by direct computation, or by using an orthogonality property of eigenvectors for symmetric operators with different eigenvalues. The symmetry of \( u \to u'' \) via integration by parts needs the boundary conditions to succeed: with \( v(0) = v(2\pi) \),

\[
\langle u'', v \rangle = \int_0^{2\pi} u''(x) \cdot \overline{v(x)} \, dx = \left[ u'(x) \overline{v(x)} \right]_0^{2\pi} - \int_0^{2\pi} u'(x) \cdot \overline{v'}(x) \, dx = -\int_0^{2\pi} u'(x) \cdot \overline{v'}(x) \, dx
\]

Symmetrically, with \( u(0) = u(2\pi) \), this is also equal to \( \langle u, v'' \rangle \). Thus, at least heuristically, on general principles the eigenvalues of \( Tu = u'' \) are real: for \( u \) a non-zero \( \lambda \)-eigenvector,

\[
\lambda \langle u, u \rangle = \langle Tu, u \rangle = \langle u, Tu \rangle = \langle u, \lambda u \rangle = \overline{\lambda} \cdot \langle u, u \rangle
\]
For \( \lambda \neq \mu \) and corresponding eigenvalues \( u, v \),
\[
\lambda \cdot \langle u, v \rangle = \langle Tu, v \rangle = \langle u, Tv \rangle = \langle u, \mu v \rangle = \mu \cdot \langle u, v \rangle
\]
Thus, \( \langle u, v \rangle = 0 \). A potential problem in this proof of orthogonality is that \( u \to u'' \) is not exactly a map \( L^2[0, 2\pi] \to L^2[0, 2\pi] \). We need not worry about this for the moment, since we can also prove the orthogonality by direct computation.

The immediate question is of completeness: are the functions \( u_n(x) = \sin \frac{n}{2}x \) an orthogonal Hilbert space basis for \( L^2[0, 2\pi] \)?

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### 2. The Green’s function

In one dimension, as we are here, the idea is to solve the differential equation \( \Delta u = f \) with boundary-value conditions \( u(0) = 0 = u(2\pi) \) by finding a Green’s function \( G(x, y) \) on \([0, 2\pi] \times [0, 2\pi] \), that is, such that
\[
\frac{d^2}{dx^2} \int_0^{2\pi} G(x, y) f(y) \, dy = f(x)
\]
for \( f \) in \( L^2[0, 2\pi] \) and meeting the boundary conditions (in a suitable sense).

In particular, when \( G(x, y) \) is in \( L^2([0, 2\pi] \times [0, 2\pi]) \), then the operator
\[
f \to \int_0^{2\pi} G(x, y) f(y) \, dy
\]
is Hilbert-Schmidt, hence compact. When also \( G(y, x) = \overline{G(x, y)} \), the operator is self-adjoint, and the spectral theorem applies. That is, the topological closure of the algebraic span of the eigenvectors is the whole \( L^2[0, 2\pi] \). That is, the eigenvectors are complete.

The spectral theorem has further important parts: eigenspaces for distinct eigenvalues are mutually orthogonal, eigenspaces for non-zero eigenvalues are finite-dimensional, and there are only finitely-many eigenvalues larger than a given bound.

In the case at hand, following the general Sturm-Liouville prescription for Green’s functions, we want \( G(x, y) \) to be annihilated by \( \frac{d^2}{dx^2} \) away from \( 0, y, 2\pi \), \( G(0, y) = 0 = G(2\pi, y) \), and \( G(x, y) \) is continuous at \( x = y \). The last requirement is that the difference in slopes of \( x \to G(x, y) \) to the left of \( y \) increases by 1 to the right of \( y \). Here, the part of \( G \) in \( 0 < x < y \) is of the form \( x \to ax \), in in \( 2\pi > x > y \) is \( x \to b(2\pi - x) \), and the continuity and slope-difference conditions give
\[
G(x, y) = \begin{cases} 
  x \cdot \left( \frac{y}{2\pi} - 1 \right) & \text{(in } 0 < x < y) \\
  y \cdot \left( \frac{x}{2\pi} - 1 \right) & \text{(in } 2\pi > x > y) 
\end{cases}
\]
From more modern viewpoint, the previous function satisfies
\[
\frac{d^2}{dx^2} G(x, y) = \delta_{x-y} - \delta_0
\]
And we should have extended \( G(x, y) \) by \( 2\pi \)-periodicity, or, equivalently, look at \([0, 2\pi] \) as parametrizing the circle \( T = \mathbb{R}/2\pi\mathbb{Z} \), to avoid counting the endpoints twice in computations.

Thus, assuming we can more the differential operator inside the integral, and writing integrals involving \( \delta \) functions when we should really be writing pairings on Sobolev spaces \( H^s(T) \times H^{-s}(T) \),
\[
\frac{d^2}{dx^2} \int_T G(x, y) f(y) \, dy = \int_T \frac{d^2}{dx^2} G(x, y) f(y) \, dy = \int_T (\delta_{x-y} - \delta_0) f(y) \, dy = f(x) - f(0)
\]
For $f(0) = 0$ in a suitable sense, this shows that this $G(x,y)$ does solve the boundary-value problem. The solution map is compact and self-adjoint, so there is an orthogonal basis of eigenfunctions. Applying $d^2/dx^2$ to an eigenvalue equation

$$
\int_T G(x,y) u(y) \, dy = \mu \cdot u(x)
$$

gives

$$
u - u(0) = \mu \cdot u''
$$

With $u(0) = 0$, and taking $\lambda = \mu^{-1}$, we recover the differential equation explicitly solved earlier by elementary methods.

Thus, the functions $\sin \frac{nx}{T}$ give an orthogonal basis for $L^2[0, 2\pi]$.

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