1. Variant Green’s function

A discussion of Fourier series as a consequence of the spectral theorem for self-adjoint compact operators on Hilbert spaces is anachronistic. Nevertheless, it does offer some insights.

As we may know, essentially the only general device to prove that a collection \{u_n\} of functions on an interval \([a, b]\) is an orthogonal basis for \(L^2[a, b]\) is to find a self-adjoint compact operator \(T\) on \(L^2[a, b]\) such that the eigenvectors of \(T\) are the functions \(u_n\). The same applies to most Hilbert spaces of functions.

The explicit Fourier-Dirichlet and Fejer arguments for ordinary Fourier series do not easily generalize, for more than one reason. As we know, one family of natural extensions is to Sturm-Liouville boundary-value problems (1830s), where we already see the necessity of the spectral theory of self-adjoint compact operators on Hilbert spaces, even though such ideas and proofs had to wait almost 60 years (1890s), for Bocher and Steklov, and then Hilbert and Schmidt and others a few years later.

The simplest Green’s function prescription for Sturm-Liouville problems does not immediately apply to Fourier series, that is, to the fact that exponentials \(\{x \to e^{inx} : n \in \mathbb{Z}\}\), or trigonometric functions \(\{1, \sin nx, \cos nx : n = 1, 2, 3, \ldots\}\), form orthogonal bases for \(L^2[0, 2\pi]\). But a slightly abstracted version does, again using the spectral theory of compact self-adjoint operators and some distribution theory, proves that the exponentials or trigonometric functions give orthogonal bases.

We want \(k\) on \(\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}\) such that

\[
\Delta k = \delta + (\text{something innocuous}) \quad (\delta = \delta_{\mathbb{Z}} \in \mathcal{D}(\mathbb{T})^*)
\]

where, ideally, the leftover is in \(L^2(\mathbb{T})\) or better. Then we would make a Hilbert-Schmidt-Schwartz kernel

\[
K(x, y) = k(x - y) \quad \text{(for } x, y \in \mathbb{T})
\]

Among other possibilities, since application of \(\Delta\) to piecewise quadratic functions, up to a constant to be determined subsequently, we take \(k(x) = -(x - \pi)^2\) on \([0, 2\pi]\) and extend by \(2\pi\mathbb{Z}\)-periodicity:

\[
\Delta k(x) = \frac{d}{dx}(-2(x - \pi))(\text{at first on } [0, 2\pi], \text{ then periodicized}) = -2 + 4\pi\delta
\]

since the jump on this sawtooth function is upward by \(2\pi\). Thus, replacing \(k(x)\) by \(k(x)/4\pi\),

\[
\Delta k = \delta - \frac{1}{2\pi}
\]

Thus, with \(K(x, y) = k(x - y)\), and

\[
Tf(x) = \int_{\mathbb{T}} K(x, y) f(y) \, dy \quad \text{(for suitable } f \in L^2(\mathbb{T}))
\]
presumably moving the differential operator inside the integral via Gelfand-Pettis, and abusing notation toward the end,

$$\Delta T f(x) = \Delta \int_T K(x, y) f(y) \, dy = \int_T \Delta_x K(x, y) f(y) \, dy = \int_T \left( \delta(x-y) - \frac{1}{2\pi} \right) f(y) \, dy = f(x) - \frac{1}{2\pi} \int_T f$$

This computation is a proof when the integral is rewritten as a pairing among Sobolev spaces. This strongly suggests that the target \( f \in L^2(\mathbb{T}) \) should satisfy \( \int_T f = 0 \).

So we consider the Hilbert space \( V = \{ u \in L^2(\mathbb{T}) : f_T u = 0 \} \). The kernel \( K(x, y) \) should be further adjusted, if necessary, so that \( x \in K(x, y) \) is in \( V \) for every \( y \). Compute

$$\int_T K(x, y) \, dx = \int_T k(x-y) \, dx = \int_T k(x) \, dx = -\frac{1}{4\pi} \int_0^{2\pi} (x-\pi)^2 \, dx = -\frac{1}{4\pi} \cdot \frac{2\pi^3}{3} = -\frac{\pi^2}{6}$$

Thus, we should add \( -\frac{x^2}{8} \cdot 2\pi \) to \( K(x, y) \). This has no impact when \( K(x, y) \) is integrated against \( f \in V \).

Since \( K(x, y) \) is continuous on \( \mathbb{T} \times \mathbb{T} \), it is in \( L^2(\mathbb{T} \times \mathbb{T}) \), and gives a Hilbert-Schmidt operator. The function \( k(x) \) itself is even and real-valued, so \( K(x, y) \) is a hermitian kernel, and gives a self-adjoint compact operator. Thus, by the spectral theorem, its eigenvectors give an orthogonal basis for \( V \).

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2. Eigenfunctions

The eigenfunction condition \( Tu = \lambda \cdot u \) for \( u \in V \) with \( \lambda \neq 0 \) implies

$$u(x) = \frac{1}{\lambda} \int_T K(x, y) u(y) \, dy \quad \text{(in } L^2(\mathbb{T}))$$

Applying \( \Delta \) distributionally, abusing notation about pairings among Sobolev spaces as usual, by design

$$\Delta u(x) = \frac{1}{\lambda} \Delta \int_T K(x, y) u(y) \, dy = \frac{1}{\lambda} \int_T \Delta_x K(x, y) u(y) \, dy = \frac{1}{\lambda} \int_T \delta(x-y) u(y) \, dy = \frac{1}{\lambda} \lambda u(x)$$

Thus, a \( \lambda \)-eigenfunction \( u \in L^2(\mathbb{T}) \) satisfies the distributional differential equation \( u'' = \frac{1}{\lambda} u \). Lifting this back to \( \mathbb{R} \) via the projection \( \mathbb{R} \rightarrow \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z} \), letting \( \lambda = e^x \), the differential equation has two linearly independent solutions, \( e^{\pm cx} \).

The requirement that a linear combination \( u(x) = ae^{cx} + be^{-cx} \) is orthogonal to constants is

$$0 = \frac{ae^{2cx} - a}{c} + \frac{be^{-2cx} - b}{-c}$$

which gives

$$a \cdot (e^{2\pi c} - 1) = b \cdot (e^{-2\pi c} - 1)$$

The periodicity condition is

$$ae^{c(x+2\pi n)} + be^{-c(x+2\pi n)} = ae^{cx} + be^{-cx} \quad \text{(for all } x \in \mathbb{R}, \text{ for all } n \in \mathbb{Z})$$

At \( x = 0 \) and \( n = 1 \), this is

$$ae^{2\pi c} + be^{-2\pi c} = a + b$$

or

$$a \cdot (e^{2\pi c} - 1) = -b \cdot (e^{-2\pi c} - 1)$$

For \( a, b \) not both 0, the latter equation and the orthogonality condition give \( e^{2\pi c} = 1 \), and then \( a, b \) can be arbitrary. Conversely, when \( e^{2\pi c} = 1 \), all the conditions are met. Thus, the non-zero eigenvalues are \(-n^2\) with \( n \in \mathbb{Z} \), with corresponding eigenspaces spanned by \( e^{\pm inx} \), and these eigenvalues give an orthogonal basis for \( V \).

(For \( \lambda = 0 \), apply \( \Delta \) distributionally to \( Tu = 0 \cdot u = 0 \) to obtain \( u(x) = 0 \) for \( u \) orthogonal to constants.)