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10a. Schwartz kernel theorems [draft]

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Hilbert-Schmidt operators \(T : L^2(\mathbb{R}^m) \to L^2(\mathbb{R}^n)\) are exactly those continuous linear operators given by\(^1\)

\[
T f(y) = \int_{\mathbb{R}^m} K(x,y) f(x) \, dx
\]

with Schwartz kernels \(K(x,y) \in L^2(\mathbb{R}^{m+n})\). But most continuous linear maps \(T : L^2(\mathbb{R}^m) \to L^2(\mathbb{R}^n)\) are not Hilbert-Schmidt, so do not have Schwartz kernels in \(L^2(\mathbb{R}^{m+n})\). The obstacle is not just the non-compactness of \(\mathbb{R}\), as most continuous \(T : L^2(\mathbb{T}^m) \to L^2(\mathbb{T}^n)\) do not have kernels in this sense, either. That is, for most such \(T\) there is no \(K(x,y) \in L^2(\mathbb{T}^{m+n})\) such that

\[
T f(y) = \int_{\mathbb{T}^m} K(x,y) f(x) \, dx
\]

As it happens, enlarging the class of possible Schwartz kernels \(K(x,y)\) on \(\mathbb{R}^{m+n}\) to make every continuous linear map \(L^2(\mathbb{T}^m) \to L^2(\mathbb{T}^n)\) have a kernel goes hand-in-hand with shrinking the source \(L^2(\mathbb{T}^m)\) to test functions and enlarging the target \(L^2(\mathbb{T}^n)\) to distributions.

1. **Concrete Schwartz’ kernel theorems: statements**

Perhaps the simplest instance of a Schwartz kernel theorem is

\[\text{[1.1] Theorem:} \quad \text{Every continuous linear map } T : D(\mathbb{T}^m) \to D(\mathbb{T}^n)^* \text{ is given by a Schwartz kernel } K \in D(\mathbb{T}^{m+n})^*, \text{ by} \]

\[
(Tf)(\varphi) = K(f \otimes \varphi)
\]

where \((f \otimes \varphi)(x,y) = f(x) \cdot \varphi(y)\) for \(x \in \mathbb{T}^m\) and \(y \in \mathbb{T}^n\). And conversely. (Proof below.)

In contrast to \(\mathbb{T}^m\), where test functions and Schwartz functions and smooth functions and Sobolev spaces \(H^\infty\) are all the same, the non-compactness of \(\mathbb{R}\) causes a bifurcation: test functions and distributions, or

\[\text{[1] Yes, this use of kernel is in conflict with the use of ker } T \text{ for } T : X \to Y \text{ to denote ker } T = \{x \in X : tx = 0\}. \]

Nothing to be done about it, except possibly to prepend Schwartz or integral to the word in the present context. For that matter, integral does also have some unrelated algebraic senses, so perhaps Schwartz kernel is the best disambiguation.
Schwartz functions and tempered distributions. Both the following are tangible instances of a Schwartz kernel theorem:

[1.2] Theorem: Every continuous linear map \( T : \mathcal{D}(\mathbb{R}^m) \to \mathcal{D}(\mathbb{R}^n)^* \) is given by a Schwartz kernel \( K \in \mathcal{D}(\mathbb{R}^{m+n})^* \), by

\[
(Tf)(\varphi) = K(f \otimes \varphi)
\]

where \( (f \otimes \varphi)(x, y) = f(x) \cdot \varphi(y) \) for \( x \in \mathbb{R}^m \) and \( y \in \mathbb{R}^n \). And conversely. (Proof below.)

[1.3] Theorem: Every continuous linear map \( T : \mathcal{S}(\mathbb{R}^m) \to \mathcal{S}(\mathbb{R}^n)^* \) is given by a Schwartz kernel \( K \in \mathcal{S}(\mathbb{R}^{m+n})^* \), by

\[
(Tf)(\varphi) = K(f \otimes \varphi)
\]

where \( (f \otimes \varphi)(x, y) = f(x) \cdot \varphi(y) \) for \( x \in \mathbb{R}^m \) and \( y \in \mathbb{R}^n \). And conversely. (Proof postponed.)

The proofs depend on existence of categorically genuine tensor products of suitable topological vector spaces, such as spaces of test functions and Schwartz functions, examples of nuclear spaces, clarified below.

The nuclearity/tensor products for \( \mathcal{S}(\mathbb{R}^n) \) are more complicated than for \( \mathcal{D}(\mathbb{T}^n) \) and \( \mathcal{D}(\mathbb{R}^n) \), as it happens.

## 2. Examples of Schwartz kernels

\( \ldots \text{iou} \ldots \)

\( \delta(x-y) \), Hilbert transform, Fourier, \ldots traces \ldots !??!

## 3. Cartan-Eilenberg/Curry/Schönfinkel/Frege adjunction

An adjunction is a natural isomorphism between two related spaces of homomorphisms, of the form

\[
\text{Hom}_\mathcal{C}(LX,Y) \approx \text{Hom}_\mathcal{D}(X,RY) \quad \text{(for all } X \in \mathcal{D} \text{ and } Y \in \mathcal{C})
\]

where \( X, Y \) are objects categories \( \mathcal{D}, \mathcal{C} \), and \( L : \mathcal{D} \to \mathcal{C} \) and \( R : \mathcal{C} \to \mathcal{D} \) are functors. As suggested by the notation, \( L \) is the left adjoint and \( R \) is the right adjoint.

For \( \mathcal{C}, \mathcal{D} \) both the category of abelian groups, for example, a basic adjunction is\(^2\)

\[
\text{Hom}(A \otimes B, C) \approx \text{Hom}(A, \text{Hom}(B,C)) \quad \text{(for abelian groups } A, B, C)
\]

by

\[
\Phi \to \varphi_\Phi \text{ with } \varphi_\Phi(a)(b) = \Phi(a \otimes b) \quad \text{and} \quad \Phi_\varphi \longleftarrow \varphi \text{ with } \Phi_\varphi(a \otimes b) = \varphi(a)(b)
\]

Quantifying over \( A, C \), for fixed \( B \), this asserts that the functor \( LA = A \otimes B \) is left adjoint to \( RC = \text{Hom}(B,C) \), and \( \text{Hom}(B,-) \) is right adjoint to \( - \otimes B \).

One direction of the isomorphism is easy, namely, \( \Phi \to \varphi_\Phi \) with \( \varphi_\Phi(a)(b) = \Phi(a \otimes b) \). The other direction of the isomorphism, \( \Phi_\varphi \longleftarrow \varphi \), needs properties of the tensor product. Specifically, \( B_\varphi(a \times b) = (\varphi(a))(b) \) makes immediate sense, but we need representability of this map, in the sense that all bilinear maps \( B_\varphi : A \times B \to C \) should produce linear maps \( \Phi_\varphi \) from an object \( A \otimes B \) not depending on \( C \) or \( \varphi \).

\(^2\) Such isomorphisms have a long history. In a homological setting, they arise in Cartan-Eilenberg in the early 1950’s. In computation theory and logic, it was in H. Curry’s 1930 work (from 1924 work of M. Schönfinkel), and was visible in G. Frege’s 1895 thesis.
A suitable form of this isomorphism will prove an abstract form of a Schwartz kernel theorem, below.

In a category $\mathcal{C}$ whose objects that admit linear maps, a tensor product $X \otimes_{\mathcal{C}} Y \in \mathcal{C}$ (if it exists!) is an object with a fixed bilinear map $\tau: X \times Y \to X \otimes_{\mathcal{C}} Y$ such that, for every bilinear $X \times Y \to Z$, there is a unique linear $B: X \otimes_{\mathcal{C}} Y \to Z$ giving a commutative diagram

$$
\begin{array}{c}
\tau \quad \exists! \quad B \\
\downarrow \quad \quad \quad \downarrow
\end{array}
\begin{array}{c}
X \otimes_{\mathcal{C}} Y \\
X \times Y \to Z
\end{array}
$$

Usually, the bilinear map $X \times Y \to X \otimes_{\mathcal{C}} Y$ is not explicitly named, but/and the image of $x \times y$ in $X \otimes_{\mathcal{C}} Y$ is denoted $x \otimes y$. This is exactly the meaning of the symbols $x \otimes y$. Existence of the tensor product asserts that for given bilinear map $\beta(x \times y)$ on $X \times Y$ there is exactly one bilinear map $B$ with $B(x \otimes y) = \beta(x \times y)$.

Proof of Schwartz kernel theorems requires existence of genuine it tensor products for suitable objects in an appropriate category of topological vector spaces. For $\mathbb{C}$-vectorspaces without topologies, with the usual (algebraic) tensor product of $\mathbb{C}$-vector spaces, the adjunction is

$$\operatorname{Hom}(A \otimes \mathbb{C} \ B, C) \cong \operatorname{Hom}(A, \operatorname{Hom}_{\mathbb{C}}(B, C))$$

and the special case $C = k$ gives

$$(A \otimes_k B)^* = \operatorname{Hom}_k(A \otimes_k B, k) \cong \operatorname{Hom}_k(A, B^*) \quad (k\text{-vectorspaces } A, B, C)$$

That is, maps from $A$ to $B^*$ are given by kernels in $(A \otimes B)^*$. The validity of this adjunction for suitable topological vector spaces, and existence of genuine tensor products, requires more. As a cautionary point, we recall in an appendix the demonstration that infinite-dimensional Hilbert spaces do not have tensor products, despite constructions that may appear to produce them.

A categorically genuine tensor product of topological vector spaces $X, Y$ would be a topological vector space $X \otimes_{??} Y$ and continuous bilinear map $\tau: X \times Y \to X \otimes_{??} Y$ such that, for every continuous bilinear $\beta: X \times Y \to Z$ for some class of topological vector spaces $Z$, there is a unique continuous linear $B: X \otimes_{??} Y \to Z$ fitting into the commutative diagram as above, namely,

$$
\begin{array}{c}
\tau \quad \exists! \quad B \\
\downarrow \quad \downarrow
\end{array}
\begin{array}{c}
X \otimes_{??} Y \\
X \times Y \to Z
\end{array}
$$

Already there is an ambiguity about the continuity requirement on bilinear maps: joint continuity, or mere separate continuity? In parallel with that ambiguity, spaces of continuous linear maps $\operatorname{Hom}(Y, Z)$, and dual spaces $Y^*$ and $(X \otimes Y)^*$, have several possible topologies. A weak topology on $\operatorname{Hom}(X, Y)$ is the finite-to-open topology, which has a basis at $0$ given by sets of the form

$$U_{S,N} = \{ S \in \operatorname{Hom}(X, Y) \colon TS \subset N \} \quad \text{(for finite } S \subset X \text{ and open } N \ni 0 \text{ in } Y)$$

A strong topology on $\operatorname{Hom}(X, Y)$ is the bounded-to-open topology, which has a basis at $0$ given by sets of the form

$$U_{S,N} = \{ T \in \operatorname{Hom}(X, Y) \colon TS \subset N \} \quad \text{(for bounded } S \subset X \text{ and open } N \ni 0 \text{ in } Y)$$

Fortunately, in the simplest situations part of this ambiguity is eliminated by

**[3.1] Theorem:** For Fréchet spaces $X, Y$ and locally convex topological vector space $Z$, a separately continuous bilinear map $X \times Y \to Z$ is necessarily jointly continuous. (*Proof in an appendix.*)
Corollary: For Fréchet spaces $X,Y$, if a tensor product $X \otimes Y$ exists representing jointly continuous bilinear maps $X \times Y \to Z$, then the same object $X \otimes Y$ represents separately continuous bilinear maps $X \times Y \to Z$. That is, if for all jointly continuous $\beta : X \times Y \to Z$ there exists unique continuous $B : X \otimes Y \to Z$ giving commutative diagram

\[
\begin{array}{cc}
A \otimes B & \exists ! B \\
X \times Y & \rightarrow & Z
\end{array}
\]
then for all separately continuous $\beta$ there exists a unique continuous $B$, as well. ///

4. Topologies on Hom($Y, Z$) and continuity of bilinear maps

Logically prior to the question of existence of tensor products, which indeed do not exist for most topological vectorspaces (regardless of ambient category), is the relation between topological requirements on bilinear maps, and topologies on spaces of continuous linear maps Hom($Y, Z$). The weak finite-to-open topology and the strong bounded-to-open topology from the previous section have a role.

Claim: The collection of separately continuous bilinear maps $\beta : X \times Y \to Z$ is in bijection with $B \in \text{Hom}(X, \text{Hom}(Y, Z))$ where Hom($Y, Z$) is given the finite-to-open topology, by

$$
\beta \rightarrow B_\beta \quad \text{by} \quad B_\beta(x)(y) = \beta(x \times y)
$$

and

$$
\beta_\nu \leftarrow B \quad \text{by} \quad \beta_\nu(x \times y) = B(x)(y)
$$

Proof: The formulas are those from the Cartan-Eilenberg (et al) adjunction. The issue is topological.

Let $\beta : X \times Y \to Z$ be separately continuous. To be clear, for each $x_o \in X$, the linear map $y \to \beta(x_o, y)$ is continuous on $Y$, and symmetrically. Then $B_\beta(x)(y) = \beta(x \times y)$

///

Claim: The collection of jointly continuous bilinear maps $\beta : X \times Y \to Z$ is in bijection with $\text{Hom}(X, \text{Hom}(Y, Z))$ where Hom($Y, Z$) is given the finite-to-open topology.

Proof:

///

5. Hilbert-Schmidt operators

We recall some features of Hilbert-Schmidt operators.

Prototype: integral operators

For $K(x, y)$ in $C^o([a, b] \times [a, b])$, define $T : L^2[a, b] \to L^2[a, b]$ by

$$
Tf(y) = \int_a^b K(x, y) f(x) \, dx
$$
The function $K$ is the integral kernel, or Schwartz kernel of $T$. Approximating $K$ by finite linear combinations of 0-or-1-valued functions shows $T$ is a uniform operator norm limit of finite-rank operators, so is compact. The Hilbert-Schmidt operators include such operators, where the integral kernel $K(x,y)$ is allowed to be in $L^2([a,b] \times [a,b])$.

[5.2] Hilbert-Schmidt norm on $V \otimes_{\text{alg}} W$

In the category of Hilbert spaces and continuous linear maps, there is no tensor product in the categorical sense, as demonstrated in an appendix.

Without claiming anything about genuine tensor products in any category of topological vector spaces, the algebraic tensor product $X \otimes_{\text{alg}} Y$ of two Hilbert spaces has a hermitian inner product $(\cdot,\cdot)_{\text{HS}}$ determined by

$$\langle x \otimes y, x' \otimes y' \rangle_{\text{HS}} = \langle x, x' \rangle \langle y, y' \rangle$$

Let $X \otimes_{\text{HS}} Y$ be the completion with respect to the corresponding norm $|v|_{\text{HS}} = (v,v)_{\text{HS}}^{1/2}$

$$X \otimes_{\text{HS}} Y = |\cdot|_{\text{HS}}\text{-completion of } X \otimes_{\text{alg}} Y$$

This completion is a Hilbert space. Unfortunately, it is not a genuine tensor product of $X,Y$, when both are infinite-dimensional, in effect because not every continuous linear map $X \to Y^*$ is Hilbert-Schmidt.

[5.3] Hilbert-Schmidt operators

For Hilbert spaces $V,W$ the finite-rank[3] continuous linear maps $T : V \to W$ can be identified with the algebraic tensor product $V^* \otimes_{\text{alg}} W$, by[4]

$$(\lambda \otimes w)(v) = \lambda(v) \cdot w$$

The space of Hilbert-Schmidt operators $V \to W$ is the completion of the space $V^* \otimes_{\text{alg}} W$ of finite-rank operators, with respect to the Hilbert-Schmidt norm $|\cdot|_{\text{HS}}$ on $V^* \otimes_{\text{alg}} W$. For example,

$$|\lambda \otimes w + \lambda' \otimes w'|_{\text{HS}}^2 = (\lambda \otimes w + \lambda' \otimes w', \lambda \otimes w + \lambda' \otimes w')$$

$$= \langle \lambda \otimes w, \lambda \otimes w \rangle + \langle \lambda \otimes w, \lambda' \otimes w' \rangle + \langle \lambda' \otimes w', \lambda \otimes w \rangle + \langle \lambda' \otimes w', \lambda' \otimes w' \rangle$$

$$= |\lambda|^2 |w|^2 + \langle \lambda, \lambda' \rangle \langle w, w' \rangle + \langle \lambda', \lambda \rangle \langle w', w \rangle + |\lambda'|^2 |w'|^2$$

When $\lambda \perp \lambda'$ or $w \perp w'$, the monomials $\lambda \otimes w$ and $\lambda' \otimes w'$ are orthogonal, and

$$|\lambda \otimes w + \lambda' \otimes w'|_{\text{HS}}^2 = |\lambda|^2 |w|^2 + |\lambda'|^2 |w'|^2$$

That is, the space $\text{Hom}_{\text{HS}}(V,W)$ of Hilbert-Schmidt operators $V \to W$ is the closure of the space of finite-rank maps $V \to W$, in the space of all continuous linear maps $V \to W$, under the Hilbert-Schmidt norm. By construction, $\text{Hom}_{\text{HS}}(V,W)$ is a Hilbert space.

[5.4] Expressions for Hilbert-Schmidt norm, adjoints

[3] As usual a finite-rank linear map $T : V \to W$ is one with finite-dimensional image.

[4] Proof of this identification: on one hand, a map coming from $V^* \otimes_{\text{alg}} W$ is a finite sum $\sum_i \lambda_i \otimes w_i$, so certainly has finite-dimensional image. On the other hand, given $T : V \to W$ with finite-dimensional image, take $v_1, \ldots, v_n$ to be an orthonormal basis for the orthogonal complement $(\ker T)^\perp$ of $\ker T$. Define $\lambda_i \in V^*$ by $\lambda_i(v) = \langle v, v_i \rangle$. Then $T \sim \sum_i \lambda_i \otimes Tv_i$ is in $V^* \otimes W$. The second part of the argument uses the completeness of $V$. 

5
The Hilbert-Schmidt norm of finite-rank $T : V \to W$ can be computed from any choice of orthonormal basis $v_i$ for $V$, by

$$|T|_{\text{HS}}^2 = \sum_i |Tv_i|^2 \quad (\text{at least for finite-rank } T)$$

Thus, taking a limit, the same formula computes the Hilbert-Schmidt norm of $T$ known to be Hilbert-Schmidt. Similarly, for two Hilbert-Schmidt operators $S, T : V \to W$,

$$\langle S, T \rangle_{\text{HS}} = \sum_i \langle S v_i, T v_i \rangle \quad (\text{for any orthonormal basis } v_i)$$

The Hilbert-Schmidt norm $|\cdot|_{\text{HS}}$ dominates the uniform operator norm $|\cdot|_{\text{op}}$: given $\varepsilon > 0$, take $|v_1| \leq 1$ with $|Tv_1|^2 + \varepsilon > |T|^2_{\text{HS}}$. Choose $v_2, v_3, \ldots$ so that $v_1, v_2, \ldots$ is an orthonormal basis. Then

$$|T|_{\text{op}}^2 \leq |Tv_1|^2 + \varepsilon \leq \varepsilon + \sum_n |Tv_n|^2 = \varepsilon + |T|_{\text{HS}}^2$$

This holds for every $\varepsilon > 0$, so $|T|_{\text{op}}^2 \leq |T|_{\text{HS}}^2$. Thus, Hilbert-Schmidt limits are operator-norm limits, and Hilbert-Schmidt limits of finite-rank operators are compact.

**Adjoints** $T^* : W \to V$ of Hilbert-Schmidt operators $T : V \to W$ are Hilbert-Schmidt, since for an orthonormal basis $w_j$ of $W$

$$\sum_i |Tv_i|^2 = \sum_{ij} |\langle v_i, w_j \rangle|^2 = \sum_{ij} |\langle v_i, T^* w_j \rangle|^2 = \sum_j |T^* w_j|^2$$

[5.5] **Criterion for Hilbert-Schmidt operators**

We claim that a continuous linear map $T : V \to W$ with Hilbert space $V$ is Hilbert-Schmidt if for some orthonormal basis $v_i$ of $V$

$$\sum_i |Tv_i|^2 < \infty$$

and then (as above) that sum computes $|T|_{\text{HS}}^2$. Indeed, given that inequality, letting $\lambda_i(v) = \langle v, v_i \rangle$, $T$ is Hilbert-Schmidt because it is the Hilbert-Schmidt limit of the finite-rank operators

$$T_n = \sum_{i=1}^n \lambda_i \otimes Tv_i$$

[5.6] **Composition of Hilbert-Schmidt operators with continuous operators**

Post-composing: for Hilbert-Schmidt $T : V \to W$ and continuous $S : W \to X$, the composite $S \circ T : V \to X$ is Hilbert-Schmidt, because for an orthonormal basis $v_i$ of $V$,

$$\sum_i |S \circ Tv_i|^2 \leq \sum_i |S|_{\text{op}}^2 \cdot |Tv_i|^2 = |S|_{\text{op}} \cdot |T|_{\text{HS}}^2 \quad (\text{with operator norm } |S|_{\text{op}} = \sup_{|v| \leq 1} |Sv|)$$

Pre-composing: for continuous $S : X \to V$ with Hilbert $X$ and orthonormal basis $x_j$ of $X$, since adjoints of Hilbert-Schmidt are Hilbert-Schmidt, 

$$T \circ S = (S^* \circ T^*)^* = (\text{Hilbert-Schmidt})^* = \text{Hilbert-Schmidt}$$

6. **Nuclear Fréchet spaces**
Roughly, the intention of \textit{nuclear spaces} is that they should admit genuine \textit{tensor products}, aiming at an abstract Schwartz Kernel Theorem.

Enlarging the class of possible Schwartz kernels $K(x,y)$ sufficiently so that every continuous $L^2(T^n) \to L^2(T^n)$ has such a kernel turns requires a larger family of topological vector spaces than Hilbert spaces or Banach spaces, so that some of them have \textit{tensor products}.

Countable projective limits of Hilbert spaces with Hilbert-Schmidt transition maps constitute the simplest class of \textit{nuclear spaces}: they admit \textit{tensor products}, as we see below. Countable limits of Hilbert spaces are also Fréchet, so these are \textit{nuclear Fréchet} spaces.

The simplest natural example of such a space is the Levi-Sobolev space $H$, also Fréchet, so these are for chosen orthonormal bases $v$ a constant $C$.

\textbf{Proof:} Let $\beta(v,\lambda) = \lambda(v)$ already illustrates this point, since not every Hilbert-Schmidt operator has a trace. That is, letting $v_i$ be an orthonormal basis for $V$ and $\lambda_i(v) = \langle v,v_i \rangle$ an orthonormal basis for $V^*$, necessarily

\[ B(\sum_{ij} c_{ij} v_i \otimes \lambda_j) = \sum_{ij} c_{ij} \beta(v_i, \lambda_j) = \sum_{i} c_{ii} \]

However, $\sum_i \frac{1}{c} v_i \otimes \lambda_i$ is in $V \otimes_{HS} V^*$, but the alleged value of $B$ is impossible. In effect, the obstacle is that there are Hilbert-Schmidt maps which are not of trace class.

\textbf{6.1} $V \otimes_{HS} W$ is not a categorical tensor product

Again, the Hilbert space $V \otimes_{HS} W$ is not a categorical tensor product of (infinite-dimensional) Hilbert spaces $V, W$. In particular, although the bilinear map $V \times W \to V \otimes_{HS} W$ is continuous, there are continuous bilinear $\beta : V \times W \to X$ to Hilbert spaces $H$ which do \textit{not} factor through any continuous linear map $B : V \otimes_{HS} W \to X$.

The case $W = V^*$ and $X = \mathbb{C}$, with $\beta(v,\lambda) = \lambda(v)$ already illustrates this point, since not every Hilbert-Schmidt operator has a trace. That is, letting $v_i$ be an orthonormal basis for $V$ and $\lambda_i(v) = \langle v,v_i \rangle$ an orthonormal basis for $V^*$, necessarily

\[ B(\sum_{ij} c_{ij} v_i \otimes \lambda_j) = \sum_{ij} c_{ij} \beta(v_i, \lambda_j) = \sum_{i} c_{ii} \]

However, $\sum_i \frac{1}{c} v_i \otimes \lambda_i$ is in $V \otimes_{HS} V^*$, but the alleged value of $B$ is impossible. In effect, the obstacle is that there are Hilbert-Schmidt maps which are not of trace class.

\textbf{6.2} Approaching tensor products and nuclear spaces

Let $V, W, V_1, W_1$ be Hilbert spaces with Hilbert-Schmidt maps $S : V_1 \to V$ and $T : W_1 \to W$. We claim that for any (jointly) continuous $\beta : V \times W \to X$, there is a unique continuous $B : V_1 \otimes_{HS} W_1 \to X$ giving a commutative diagram

\[ V_1 \otimes_{HS} W_1 \xrightarrow{B} V \otimes_{HS} W \]

\[ V_1 \times W_1 \xrightarrow{S \times T} V \times W \xrightarrow{\beta} X \]

In fact, $B : V_1 \otimes_{HS} W_1 \to X$ is \textit{Hilbert-Schmidt}. As the diagram suggests, $V \otimes_{HS} W$ is bypassed, playing no role.

\textbf{Proof:} Once the assertion is formulated, the argument is the only thing it can be: The continuity of $\beta$ gives a constant $C$ such that $|\beta(v,w)| \leq C \cdot |v| \cdot |w|$, for all $v \in V, w \in W$. The Hilbert-Schmidt condition is that, for chosen orthonormal bases $v_i$ of $V_1$ and $w_j$ of $W_1$,

\[ |S|^2_{HS} = \sum_i |Sv_i|^2 < \infty \quad |T|^2_{HS} = \sum_j |Tw_j|^2 < \infty \]

Thus,

\[ |\beta(Sv,Tw)| \leq C \cdot |Sv| \cdot |Tv| \]
Squaring and summing over \( v_i \) and \( w_j \),
\[
\sum_{ij} |\beta(Sv_i,Tw_j)|^2 \leq C \cdot \sum_{ij} |Sv_i|^2 \cdot |Tw_j|^2 = C \cdot |S_{\text{HS}}|^2 \cdot |T_{\text{HS}}|^2 < \infty
\]

That is, with the obvious definition-attempt
\[
B(\sum_{ij} c_{ij} v_i \otimes w_j) = \sum_{ij} c_{ij} \beta(Sv_i,Tw_j)
\]

Cauchy-Schwarz-Bunyakowsky
\[
\sum_{ij} |c_{ij} \beta(Sv_i,Tw_j)|^2 \leq \sum_{ij} |c_{ij}|^2 \cdot \sum_{ij} |\beta(Sv_i,Tw_j)|^2 \leq \sum_{ij} |c_{ij}|^2 \cdot \left( C \cdot |S_{\text{HS}}|^2 \cdot |T_{\text{HS}}|^2 \right)
\]

shows that \( B : V_1 \otimes W_1 \to X \) is Hilbert-Schmidt.  

[6.3] A class of nuclear Fréchet spaces

We take the basic nuclear Fréchet space to be a countable limit of Hilbert spaces where the transition maps are Hilbert-Schmidt.

That is, for a countable collection of Hilbert spaces \( V_0, V_1, V_2, \ldots \) with Hilbert-Schmidt maps \( \varphi_i : V_i \to V_{i-1} \), the limit \( V = \lim_i V_i \) in the category of locally convex topological vector spaces is a nuclear Fréchet space.

Let \( \mathcal{C} \) be the category of Hilbert spaces enlarged to include limits.

[6.4] Theorem: Nuclear Fréchet spaces admit tensor products in \( \mathcal{C} \). That is, for nuclear spaces \( V = \lim_i V_i \) and \( W = \lim_i W_i \) there is a nuclear space \( V \otimes W \) and continuous bilinear \( V \times W \to V \otimes W \) such that, given a jointly continuous bilinear map \( \beta : V \times W \to X \) of nuclear spaces \( V, W \) to \( X \in \mathcal{C} \), there is a unique continuous linear map \( B : V \otimes W \to X \) giving a commutative diagram

\[
\begin{array}{ccc}
V \otimes W & \xrightarrow{B} & X \\
\uparrow & & \swarrow \\
V \times W & \xrightarrow{\beta} & X
\end{array}
\]

In particular, \( V \otimes W \approx \lim_i V_i \otimes_{\text{HS}} W_i \).

Proof: As will be seen at the end of this proof, the defining property of (projective) limits reduces to the case that \( X \) is itself a Hilbert space. Let \( \varphi_i : V_i \to V_{i-1} \) and \( \psi_i : W_i \to W_{i-1} \) be the transition maps. First, we claim that, for large-enough index \( i \), the bilinear map \( \beta : V \times W \to X \) factors through \( V_i \times W_i \). Indeed, the topologies on \( V \) and \( W \) are such that, given \( \varepsilon > 0 \), there are indices \( i, j \) and open neighborhoods of zero \( E \subset V_i, F \subset W_j \) such that \( \beta(E \times F) \subset \varepsilon \)-ball at 0 in \( X \). Since \( \beta \) is \( C \)-bilinear, for any \( \varepsilon > 0 \),

\[
\beta(\frac{\varepsilon}{\varepsilon a} E \times F) \subset \varepsilon \text{-ball at 0 in } X
\]

[5] Properly, the class of categorical limits includes products and other objects whose indexing sets are not necessarily directed. In that context, requiring that the index set be directed, a projective limit is a directed or filtered limit. Similarly, what we will call simply colimits are properly filtered or directed colimits.

[6] The new aspect is the nuclearity, not the Fréchet-ness: an arbitrary countable limit of Hilbert spaces is (provably) Fréchet, since an arbitrary countable limit of Fréchet spaces is Fréchet.

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That is, $\beta$ is already continuous in the $V_i \times W_j$ topology. Replace $i, j$ by their maximum, so $i = j$.

The argument of the previous section exhibits continuous linear $B$ fitting into the diagram

$$
\begin{array}{ccc}
V_{i+1} \otimes_{HS} W_{i+1} & \rightarrow & X \\
\uparrow & & \downarrow B \\
V_{i+1} \times W_{i+1} & \rightarrow & V_i \times W_i \\
& \beta & \\
\end{array}
$$

In fact, $B$ is Hilbert-Schmidt. Applying the same argument with $X$ replaced by $V_{i+1} \otimes_{HS} W_{i+1}$ shows that the dotted map in

$$
\begin{array}{ccc}
V_{i+2} \otimes_{HS} W_{i+2} & \rightarrow & V_{i+1} \otimes_{HS} W_{i+1} \\
\uparrow & & \downarrow B \\
V_{i+2} \times W_{i+2} & \rightarrow & V_{i+1} \times W_{i+1} \\
& \beta & \\
\end{array}
$$

is Hilbert-Schmidt. Thus, the categorical tensor product is the limit of the Hilbert-Schmidt completions of the algebraic tensor products of the limitands:

$$(\lim_i V_i) \otimes (\lim_j W_j) = \lim_i (V_i \otimes_{HS} W_i)$$

The transition maps in this limit have been proven Hilbert-Schmidt, so the limit is again nuclear.

As remarked at the beginning of the proof, the general case follows from the basic characterization of projective limits: for $X = \lim_i X_i$ with $X_i$ Hilbert, a continuous bilinear map $V \otimes W \to X$ is exactly a compatible family of maps $V \otimes W \to X_i$. To obtain this compatible family, observe that a continuous bilinear $V \times W \to X$ composed with projections $X \to X_i$ gives a compatible family of continuous bilinear maps $V \times W \to X_i$. These induce compatible linear maps $V \otimes W \to X_i$, as in the commutative diagram

These linear maps $V \otimes W \to X_i$ induce a unique continuous linear $V \otimes W \to X$.

7. Schwartz Kernel Theorem for nuclear Fréchet spaces

Let $X, Y$ be nuclear Fréchet space of the form $X = \lim_i X_i$ and $Y = \lim_i Y_i$ with Hilbert spaces $X_i, Y_i$ and Hilbert-Schmidt transition maps $X_i \to X_{i-1}$ and $Y_i \to Y_{i-1}$. We have an abstract kernel theorem:

[7.1] Theorem: $\text{Hom}_c(X, Y^*) \approx (X \otimes c Y)^*$

Proof: Given the existence of the tensor product from above, it suffices to show that the vector space

$\text{Bil}(X \times Y, \mathbb{C})$

of continuous bilinear maps is linearly isomorphic to $\text{Hom}_c(X, Y^*)$, via the expected

$$
\beta \mapsto (x \mapsto (y \mapsto \beta(x, y))) \\
$$

(for $\beta \in \text{Bil}^0(X, Y)$, $x \in X$, and $y \in Y$)

A potential issue is topological: what topology is put on $\text{Hom}(X, Y^*)$?
Given \( x \in X \), bounded \( E \subset Y \), and \( \varepsilon > 0 \), by joint continuity of \( \beta \), there are neighborhoods \( M, N \) of 0 in \( X, Y \) such that

\[
\beta(x + M, N) = \beta(x + M, N) - \beta(x, 0) \subset \varepsilon\text{-ball in } Y^*
\]

Since \( E \) is bounded, there is \( t > 0 \) such that \( tN \supset E \). Then

\[
\beta(x + m, e) - \beta(x, e) = \beta(m, e) \in \beta(M, E) \subset \beta(M, tN) \quad \text{(for } m \in M \text{ and } e \in E \text{)}
\]

This suggests replacing \( M \) by \( t^{-1}M \), so

\[
\beta(x + m, e) - \beta(x, e) = \beta(t^{-1}M, E) \subset \beta(t^{-1}M, tN) \subset \varepsilon\text{-ball in } Y^* \quad \text{(for } m \in t^{-1}M \text{ and } e \in E \text{)}
\]

That is,

\[
\beta(x + m, -) - \beta(x, -) \in U_{E, \varepsilon} \quad \text{(for } m \in t^{-1}M \text{)}
\]

This proves the continuity of the map \( X \to Y^* \) induced by \( \beta \).

Conversely, given \( \varphi : X \to Y^* \), put \( \beta(x, y) = \varphi(x)(y) \). For fixed \( x \), \( \beta(x, -) = \varphi(x) \) is continuous, by hypothesis. For fixed \( y \), \( E = \{ y \} \) is a bounded set in \( Y \), so by the continuity of \( x \to \varphi(x) \), for given \( x \) and \( \varepsilon > 0 \) there is a neighborhood \( M \) of 0 in \( X \) so that \( \varphi(x + M) - \varphi(x) \subset U_{E, \varepsilon} \). This proves that \( \beta(\cdot, y) \) is continuous. Thus, \( \beta \) is separately continuous. If there were any doubt, an appendix shows that separately continuous bilinear functions on Hilbert spaces are jointly continuous.

---

**8. \( \mathcal{D}(T^n) \) is nuclear Fréchet**

Let \( T \) be the circle \( \mathbb{R}/2\pi\mathbb{Z} \). In terms of Fourier series, for \( s \geq 0 \) the \( s \)-th \( L^2 \) Levi-Sobolev space on \( T^n \) is

\[
H^s(T^n) = \{ \sum_{\xi} c_{\xi} e^{i\xi \cdot x} \in L^2(T^n) : \sum_{\xi} |c_{\xi}|^2 \cdot (1 + |\xi|^2)^{s} < \infty \}
\]

The Levi-Sobolev imbedding theorem asserts that

\[
H^{k + \frac{m}{2}} + \varepsilon(T^n) \subset C^k(T^n) \quad \text{(for all } \varepsilon > 0 \text{)}
\]

Thus,

\[
C^\infty(T^n) = H^{+\infty}(T^n) = \lim_{s \to \infty} H^s(T^n) \approx \lim \left( \ldots \to H^2(T^n) \to H^1(T^n) \to H^0(T^n) \right)
\]

We recall a form of Rellich’s compactness lemma:

**[8.1] Theorem:** For \( s > t \), \( H^s(T^n) \to H^t(T^n) \) is Hilbert-Schmidt for \( s > t + \frac{n}{2} \).

*Proof:* [... iou ...]

---

**[8.2] Corollary:** \( H^\infty(T^n) = C^\infty(T^n) \) is nuclear Fréchet.

---

**9. \( \mathcal{D}(T^m) \otimes \mathcal{D}(T^n) \approx \mathcal{D}(T^{m+n}) \)**

**[9.1] Claim:**

\[
H^{+\infty}(T^m) \otimes \xi H^{+\infty}(T^n) \approx H^{+\infty}(T^{m+n})
\]
induced from the natural

$$(\varphi \otimes \psi)(x, y) = \varphi(x) \psi(y) \quad (\varphi \in H^+\infty(\mathbb{T}^m), \psi \in H^+\infty(\mathbb{T}^n), x \in \mathbb{T}^m, y \in \mathbb{T}^n)$$

Indeed, our construction of this tensor product is

$$H^+\infty(\mathbb{T}^m) \otimes_{\text{c}} H^+\infty(\mathbb{T}^n) = \lim_s \left( H^s(\mathbb{T}^m) \otimes_{\text{HS}} H^s(\mathbb{T}^n) \right)$$

The inequalities

$$(1 + |\xi|^2 + |\eta|^2)^2 \geq (1 + |\xi|^2)(1 + |\eta|^2) \geq 1 + |\xi|^2 + |\eta|^2 \quad (\text{for } \xi \in \mathbb{Z}^m, \eta \in \mathbb{Z}^n)$$

give

$$H^{2s}(\mathbb{T}^{m+n}) \subset H^s(\mathbb{T}^m) \otimes_{\text{HS}} H^s(\mathbb{T}^n) \subset H^s(\mathbb{T}^{m+n}) \quad (\text{for } s \geq 0)$$

The limit only depends on cofinal sublimits, so, indeed,

$$H^+\infty(\mathbb{T}^m) \otimes_{\text{c}} H^+\infty(\mathbb{T}^n) \approx H^+\infty(\mathbb{T}^{m+n})$$

Thus,

$$\text{Hom}(\mathcal{D}(\mathbb{T}^m), \mathcal{D}(\mathbb{T}^n)^*) \approx \text{Hom}(\mathcal{D}(\mathbb{T}^m) \otimes \mathcal{D}(\mathbb{T}^n), \mathbb{C}) \approx \text{Hom}(\mathcal{D}(\mathbb{T}^{m+n}), \mathbb{C}) = \mathcal{D}(\mathbb{T}^{m+n})^*$$

This completes the proof of a concrete Schwartz kernel theorem for $\mathcal{D}(\mathbb{T}^n)$, namely,

$$\text{Hom}(\mathcal{D}(\mathbb{T}^m), \mathcal{D}(\mathbb{T}^n)^*) \approx \mathcal{D}(\mathbb{T}^{m+n})^*$$

as asserted in the first section.
10. Nuclearity of $\mathcal{D}(\mathbb{R}^n)$

Although the statement of the Schwartz kernel theorem for test functions on $\mathbb{R}^n$ is identical to that for $\mathcal{D}(\mathbb{T}^n)$, the proof must be somewhat different, because $\mathcal{D}(\mathbb{R}^n)$ is not a Fréchet space. It is an LF-space, that is, a strict colimit (also called strict inductive limit) of Fréchet spaces. Thus, proof of existence of tensor products must be somewhat different.

[... iou ...]

11. Appendix: joint continuity of bilinear maps

Joint continuity of separately continuous bilinear maps on Hilbert spaces, is an easy corollary of Baire category. The result extends to Fréchet spaces with a little more work. First:

[11.1] Claim: A bilinear map $\beta : X \times Y \to Z$ on Hilbert spaces $X,Y,Z$, continuous in each variable separately, is jointly continuous.

Proof: Fix a neighborhood $N$ of 0 in $Z$. Take sequences $x_n \to x_o$ in $X$ and $y_n \to y_o$ in $Y$. For each $x \in X$, by continuity in $Y$, $\beta(x,y_n) \to \beta(x,y_o)$. Thus, for each $x \in X$, the set of values $\beta(x,y_n)$ is bounded in $Z$. The linear functionals $x \to \beta(x,y_n)$ are equicontinuous, by Banach-Steinhaus, so there is a neighborhood $U$ of 0 in $X$ so that $b_n(U) \subset N$ for all $n$. In the identity

$$\beta(x_n,y_n) - \beta(x_o,y_o) = \beta(x_n-x_o,y_n) + \beta(x_o,y_n-y_o)$$

we have $x_n - x_o \in U$ for large $n$, and $\beta(x_n-x_o,y_o) \in N$. Also, by continuity in $Y$, $\beta(x_o,y_n-y_o) \in N$ for large $n$. Thus, $\beta(x_n,y_n) - \beta(x_o,y_o) \in N + N$, proving sequential continuity. Since $X \times Y$ is metrizable, sequential continuity implies continuity. ///

For a more general result, we recall some preparatory ideas:

A set $E$ of continuous linear maps from a topological vectorspace $X$ to $Y$ is equicontinuous when, for every neighborhood $U$ of 0 in $Y$, there is a neighborhood $N$ of 0 in $X$ so that $T(N) \subset U$ for every $T \in E$.

[11.2] Claim: Let $V$ be a strict colimit of a locally convex closed subspaces $V_i$. Let $Y$ be a locally convex topological vectorspace. A set $E$ of continuous linear maps from $V$ to $Y$ is equicontinuous if and only if for each index $i$ the collection $E|_{V_i} = \{T|_{V_i} : T \in E\}$ of restrictions is equicontinuous.

Proof: Given a neighborhood $U$ of 0 in $Y$, shrink $U$ if necessary so that $U$ is convex and balanced. For each index $i$, let $N_i$ be a convex, balanced neighborhood of 0 in $V_i$ so that $TN_i \subset U$ for all $T \in E$. Let $N$ be the convex hull of the union of the $N_i$ in the locally convex coproduct of the $V_i$. By the convexity of $N$, still $TN \subset U$ for all $T \in E$. By the construction of the coproduct topology as the diamond topology, $N$ is an open neighborhood of 0 in the coproduct. Hence the image of $N$ in the colimit, a quotient of the coproduct, is a neighborhood of 0. This gives the equicontinuity of $E$. The other direction of the implication is easy. ///

Next, we need

[11.3] Claim: Banach-Steinhaus/uniform boundedness Let $X$ be a Fréchet space or LF-space and $Y$ an arbitrary topological vector space. A set $E$ of linear maps $X \to Y$, such that every set $Ex = \{Tx : T \in E\}$ of pointwise values is bounded in $Y$, is equicontinuous.

Proof: First consider $X$ Fréchet. Given a neighborhood $U$ of 0 in $Y$, let $A = \bigcap_{T \in E} T^{-1}U$. By assumption, $\bigcup_n nA = X$. By the Baire category theorem, the complete metric space $X$ is not a countable union of
nowhere dense subsets, so at least one of the closed sets \( nA \) has non-empty interior. Since (non-zero) scalar multiplication is a homeomorphism, \( A \) itself has non-empty interior, containing some \( x + N \) for a neighborhood \( N \) of 0 and \( x \in A \). For every \( T \in E \),

\[
TN \subset T\{a - x : a \in A\} \subset \{u_1 - u_2 : u_1, u_2 \in U\} = \bar{U} - \bar{U}
\]

By continuity of addition and scalar multiplication in \( Y \), given an open neighborhood \( U_o \) of 0, there is \( U \) such that \( \bar{U} - \bar{U} \subset U_o \). Thus, \( TN \subset U_o \) for every \( T \in E \), and \( E \) is equicontinuous.

For \( X = \bigcup_i X_i \) an LF-space, this argument already shows that \( E \) restricted to each \( X_i \) is equicontinuous. From the previous claim, this gives equicontinuity on the strict colimit. ///

A corollary of Banach-Steinhaus:

**[11.4] Corollary:** A separately continuous bilinear map \( \beta : X \times Y \to Z \) from Fréchet spaces \( X,Y \) to an arbitrary topological vector space \( Z \) is jointly continuous.

**Proof:** Fix an open \( N \ni 0 \) in \( Z \). Let \( x_n \to x_o \) in \( X \) and \( y_n \to y_o \) in \( Y \). For each \( x \in X \), by continuity in \( Y \), 

\[
\beta(x,y_n) \to \beta(x,y_o).
\]

Thus, for each \( x \in X \), the set of values \( \beta(x,y_n) \) is bounded in \( Z \). By Banach-Steinhaus, the linear functionals \( x \to \beta(x,y_n) \) are equicontinuous, so there is an open \( U \ni 0 \) in \( X \) so that \( b_n(U) \subset N \) for all \( n \). In the identity

\[
\beta(x_n,y_n) - \beta(x_o,y_o) = \beta(x_n - x_o, y_n) + \beta(x_o, y_n - y_o)
\]

\( x_n - x_o \in U \) for large \( n \), and \( \beta(x_n - x_o, y_o) \in N \). Similarly, by continuity in \( Y \), \( \beta(x_o, y_n - y_o) \in N \) for large \( n \). Thus, \( \beta(x_n,y_n) - \beta(x_o,y_o) \in N + N \), proving sequential continuity. Since \( X \times Y \) is metrizable, sequential continuity implies continuity. ///

**[11.5] Corollary:** The same conclusion holds for LF-spaces \( X,Y \).

**Proof:** Continuous linear functionals from an LF-space are exactly given by compatible families of continuous maps from the limitands. ///

**[11.6] Corollary:** For topological vector spaces \( X,Y \) such that sequential continuity of bilinear maps \( X \times Y \to Z \) implies full continuity, separately continuous bilinear maps are jointly continuous. ...

**[11.7] Corollary:** Separately continuous bilinear maps \( X \times Y \to Z \) are jointly sequentially continuous. ///

**[11.8] Claim:** Separately continuous bilinear maps \( B : X \times Y \to Z \) are given by continuous elements of \( \text{Hom}(X, \text{Hom}(Y,Z)) \) where \( \text{Hom}(Y,Z) \) is given a weak dual topology. Jointly continuous bilinear maps \( B : X \times Y \to Z \) are given by continuous elements of \( \text{Hom}(X, \text{Hom}(Y,Z)) \) where \( \text{Hom}(Y,Z) \) is given a strong dual topology.

**Proof:**

///

Thus, we can remove some ambiguity in characterization of a tensor product of nuclear Fréchet spaces:

**[11.9] Corollary:** For Fréchet \( X,Y \), for arbitrary (locally convex) \( Z \), the space of continuous linear maps \( \text{Hom}(X, \text{Hom}(Y,Z)) \) is the same for either the strong or weak topology on \( \text{Hom}(Y,Z) \).
12. Appendix: non-existence of tensor products of Hilbert spaces

Tensor products of infinite-dimensional Hilbert spaces do not exist.

That is, for infinite-dimensional Hilbert spaces \( V, W \), there is no Hilbert space \( X \) and continuous bilinear map \( j : V \times W \to X \) such that, for every continuous bilinear \( V \times W \to Y \) to a Hilbert space \( Y \), there is a unique continuous linear \( X \to Y \) fitting into the commutative diagram

\[
\begin{array}{ccc}
X \\
\uparrow j \\
V \times W \\ \downarrow \downarrow \downarrow \\
Y
\end{array}
\]

That is, there is no tensor product in the category of Hilbert spaces and continuous linear maps.

Yes, it is possible to put an inner product on the algebraic tensor product \( V \otimes_{\text{alg}} W \), by

\[
\langle v \otimes w, v' \otimes w' \rangle = \langle v, v' \rangle \cdot \langle w, w' \rangle
\]

and extending. The completion \( V \otimes_{\text{HS}} W \) of \( V \otimes_{\text{alg}} W \) with respect to the associated norm, is a Hilbert space, identifiable with Hilbert-Schmidt operators \( V \to W^* \). However, this Hilbert space fails to have the universal property in the categorical characterization of tensor product, as we see below. This Hilbert space \( H \) is important in its own right, but is widely misunderstood as being a tensor product in the categorical sense.

The non-existence of tensor products of infinite-dimensional Hilbert spaces is important in practice, not only as a cautionary tale\(^7\) about naive category theory, insofar as it leads to Grothendieck’s idea of nuclear spaces, which do admit tensor products.

Proof: First, we review the point that the Hilbert-Schmidt tensor product \( H = V \otimes_{\text{HS}} W \) is not a Hilbert-space tensor product, although it is a Hilbert space. For simplicity, suppose that \( V, W \) are separable, in the sense of having countable Hilbert-space bases.

Choice of such bases allows an identification of \( W \) with the continuous linear Hilbert space dual \( V^* \) of \( V \). Then we have the continuous bilinear map \( V \times V^* \to \mathbb{C} \) by \( v \times \lambda \to \lambda(v) \). The algebraic tensor product \( V \otimes_{\text{alg}} W \) injects to \( H = V \otimes_{\text{HS}} V^* \), and the image is identifiable with the finite-rank maps \( V \to V \). The linear map \( T : H \to \mathbb{C} \) induced on the image of \( V \otimes_{\text{alg}} V^* \) is trace. If \( H = V \otimes_{\text{HS}} V^* \) were a Hilbert-space tensor product, the trace map would extend continuously to it from finite-rank operators. However, there are many Hilbert-Schmidt operators that are not of trace class. For example, letting \( e_i \) be an orthonormal basis, the element

\[
\sum_n \frac{1}{n} \cdot e_n \otimes e_n \in V \otimes_{\text{HS}} V^*
\]

does not have a finite trace, since \( \sum_{n \leq N} 1/n \sim \log N \). In other words, the difficulty is that

\[
T\left( \sum_{a \leq n \leq b} \frac{1}{n} \cdot e_n \otimes e_n \right) = \sum_{a \leq n \leq b} \frac{1}{n} \cdot T(e_n \otimes e_n) = \sum_{a \leq n \leq b} \frac{1}{n}
\]

\(^7\) Many of us are not accustomed to worry about existence of objects defined by universal mapping properties, because we proved their existence by set-theoretic constructions of them, long before becoming aware of mapping-property characterizations. Much as naive set theory does not lead to paradoxes without effort, naive category theory’s recharacterization of objects close to prior experience rarely describes non-existent objects. Nevertheless, the present example is genuine.
Thus, the partial sums of $\sum_{n} \frac{1}{n} e_n \otimes e_n$ form a Cauchy sequence, but the values of $T$ on the partial sums go to $+\infty$. Thus, the Hilbert-Schmidt tensor product cannot be a Hilbert-space tensor product.

Now we show that no other Hilbert space can be a tensor product, by comparing to the Hilbert-Schmidt tensor product.

Let $V \times W \to X$ be a purported Hilbert-space tensor product, and, again, let $W$ be the dual of $V$, without loss of generality. By assumption, the continuous bilinear injection $V \times V^* \to V \otimes_{\text{HS}} V^*$ induces a unique continuous linear map $T : X \to H$ fitting into a commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{T} & V \otimes_{\text{HS}} V^* \\
\downarrow & & \downarrow \\
V \times V^* & \to & V \otimes_{\text{alg}} V^*
\end{array}
\]

The linear map $V \otimes_{\text{alg}} V^* \to V \otimes_{\text{HS}} V^*$ is injective, since $V \otimes_{\text{HS}} V^*$ is a completion of $V \otimes_{\text{alg}} V^*$. Thus, unsurprisingly, $V \otimes_{\text{alg}} V^* \to X$ is necessarily injective. The uniqueness of the linear induced maps implies that the image of $V \otimes_{\text{alg}} V^*$ is dense in $X$. Also, $T : X \to V \otimes_{\text{HS}} V^*$ is the identity on the copies of $V \otimes_{\text{alg}} V^*$ imbedded in $X$ and $V \otimes_{\text{HS}} V^*$. Let $T^* : V \otimes_{\text{HS}} V^* \to X$ be the adjoint of $T$, defined by

$$\langle x, T^* y \rangle_X = \langle Tx, y \rangle_{V \otimes_{\text{HS}} V^*}$$

On the imbedded copies of $V \otimes_{\text{alg}} V^*$

$$\langle v \otimes \lambda, T^* (w \otimes \mu) \rangle_X = \langle T(v \otimes \lambda), w \otimes \mu \rangle_{V \otimes_{\text{HS}} V^*} = \langle v \otimes \lambda, w \otimes \mu \rangle_{V \otimes_{\text{HS}} V^*} \quad \text{for} \quad v, w \in V \text{ and } \lambda, \mu \in V^*$$

Given $v \in V$ and $\lambda \in V^*$, the orthogonal complement $(v \otimes \lambda)^\perp$ is the closure of the span of monomials $v' \otimes \lambda'$ where either $v' \perp v$ or $\lambda' \perp \lambda$. For such $v' \otimes \lambda'$,

$$0 = \langle v' \otimes \lambda', v \otimes \lambda \rangle_H = \langle T(v' \otimes \lambda'), v \otimes \lambda \rangle_H = \langle v' \otimes \lambda', T^*(v \otimes \lambda) \rangle_X$$

Thus, for any monomial $v \otimes \lambda$, the image $T^*(v \otimes \lambda)$ is a scalar multiple of $v \otimes \lambda$. The same is true of monomials $(v + w) \otimes (\lambda + \mu)$. Taking $v, w$ linearly independent and $\lambda, \mu$ linearly independent and expanding shows that the scalars do not depend on $v, \lambda$. Thus, $T^*$ is a scalar on $V \otimes_{\text{alg}} V^*$.

That is, there is a (necessarily real) constant $C$ such that

$$C \cdot \langle v \otimes \lambda, w \otimes \mu \rangle_X = \langle v \otimes \lambda, T^*(w \otimes \mu) \rangle_X = \langle T(v \otimes \lambda), w \otimes \mu \rangle_{V \otimes_{\text{HS}} V^*} = \langle v \otimes \lambda, w \otimes \mu \rangle_{V \otimes_{\text{HS}} V^*}$$

since $T$ identifies the imbedded copies of $V \otimes_{\text{alg}} V^*$. That is, up to the constant $C$, the inner products from $X$ and $V \otimes_{\text{HS}} V^*$ restrict to the same hermitian form on $V \otimes_{\text{alg}} V^*$. Thus, any putative tensor product $X$ differs from $V \otimes_{\text{HS}} V^*$ only by scaling. However, we saw that the natural pairing $V \times V^* \to \mathbb{C}$ does not factor through a continuous linear map $V \otimes_{\text{HS}} V^* \to \mathbb{C}$, because there exist Hilbert-Schmidt maps not of trace class.

Thus, there is no tensor product of infinite-dimensional Hilbert spaces.
13. Appendix: closed convex hulls

For convenience and perspective about expression of Gelfand-Pettis integrals as limits of finite sums, we review some basic points.

[13.1] Claim: For a subset $E$ of a locally convex topological vector space $V$, and for continuous linear $f : V \to W$ to another locally convex topological vector space $W$, the closure of the convex hull of $f(E)$ is the closure of the convex hull of $f(E)$, where $E$ is the topological closure of $E$.

Proof: First, recall that the closure $E$ of a subset $E$ of a topological vector space is the intersection of all $E + U$ where $U$ runs over opens containing 0. Thus, the closure of the convex hull of $f(E)$ is $\bigcap f(E) + U$ with $0 \in U \subset W$.

The convex hull of $f(E)$ is the collection of finite convex combinations $\sum_{i=1}^{n} t_i f(v_i)$ with $v_i \in E$. For each $v_i$, let $x_{i,\alpha}$ be a net such that $\lim_{\alpha} x_{i,\alpha} = v_i$. The continuity of $f$ assures that $\lim_{\alpha} f(x_{i,\alpha}) = f(v_i)$. Thus, there is $\alpha_i$ such that $f(v_i) \in f(x_{i,\alpha_i}) + U$. Then

$$\sum_{i=1}^{n} t_i f(v_i) \in \sum_{i=1}^{n} t_i \cdot (f(x_{i,\alpha_i}) + U) = \left( \sum_{i=1}^{n} t_i f(x_{i,\alpha_i}) \right) + U$$

This holds for every $U \ni 0$. ///

Unsurprisingly:


Proof: In this context, boundedness is that, given an open $U \ni 0$, there is $t_o$ such that for all $z \in \mathbb{C}$ with $|z| \geq t_o$ we have $z \cdot U \supset E$. Without loss of generality, shrink $U$ so that it is balanced, in the sense that $z \cdot U \subset U$ for all $|z| \leq 1$.

First, for an individual point $v$, continuity at $z = 0$ of scalar multiplication $v \to z \cdot v$ gives that, for every open $U \ni 0$, there is $\delta > 0$ such that $z \cdot v \in U + 0 \cdot v = U$ for all $|z| < \delta$. Then $|z| > 1/\delta$ gives $v \in z \cdot U$.

For each $v \in E$, let $t_v$ be such that $v \in z \cdot U$ for all $|z| \geq t_v$. Then $E \subset \bigcup_{v \in E} t_v \cdot U$, and by compactness there are finitely-many $t_1, \ldots, t_n$ such that $E \subset t_1 U \cup \ldots \cup t_n U$. By balancedness of $U$, the latter union is contained in $t_o U$ with $t_o$ the maximum of $t_1, \ldots, t_n$. Again by balancedness, $t_o U \subset z U$ for all $|z| \geq t_o$. ///
Bibliography


