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Irreducibles as kernels of intertwinings among principal series

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Let $G = SL(2, \mathbb{R})$. Our goal is to *find* irreducible representations of G. The actual work we do is quite prosaic, and we do find several sorts of representations of G, without any fancier ideas.

However, the *completeness* of our computation depends on some more serious results. Further, the *motivation* for this computation, and the idea that it would be fruitful, come from less pedestrian thinking.

The computation itself can be understood on its own terms without understanding the broader context. For that matter, the fact that a few simple if non-elementary ideas lead to such tangible computational results might be construed as a motivation to study those non-elementary notions.

To understand the larger context, first note that Casselman's **subrepresentation theorem** asserts that any irreducible representation [1] π of G is a subrepresentation of some one of the **principal series** representations I_s defined below. [2] Further, the quotient I_s/π (or a further quotient that is irreducible) again imbeds in some $I_{s'}$ with another parameter value s'. Thus, any irreducible π appears as a (possibly subrepresentation of a) *kernel* of a G-homomorphism $I_s \to I_{s'}$ among principal series. Still further, examination of the eigenvalues of the center of the enveloping algebra [3] on the I_s shows that the only values s' such that $I_{s'}$ has a non-trivial G-homorphism $I_s \to I_{s'}$ are $s' = s$ and $s' = 1 - s$. There is a natural *integral* for a G-homomorphism $I_s \to I_{s'}$, which can be evaluated in terms of the gamma function on adroitly chosen vectors in the principal series. But, again, the computation itself is understandable without necessarily fully appreciating this grounding of it.

This computation also gives an approach to understand why the gamma function should not vanish.

[1] To be more accurate, the subrepresentation theorem in fact asserts the imbeddability of irreducible $(\mathfrak{g}, K)$ representations, also called $(\mathfrak{g}, K)$-**modules**. The aptness of this notion was one of Harish-Chandra's basic and indispensable contributions to this subject. Here $\mathfrak{g}$ is the Lie algebra of G acting by the differentiated version of the action of G (at least on *smooth* vectors of the representation), and K is a maximal compact subgroup of G. Thus, the $(\mathfrak{g}, K)$-module structure forgets some of the structure of a G-representation.

[2] Actually, our present discussion only discusses *half* the principal series, namely the *even* or *unramified* ones.

[3] The enveloping algebra is the associative algebra generated by $\mathfrak{g}$ with relations $xy - yx = [x, y]$ for $x, y \in \mathfrak{g}$. The structure of its *center* is described by an early theorem of Harish-Chandra, and by a Schur lemma the center acts by scalars on irreducibles. Further, it is our good fortune that the eigenvalues of the center quite successfully distinguish among principal series.

1. *Principal series representations*

Define useful subgroups [4] of G by

$$N = \{n_x = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{R}\} \qquad M = \{\begin{pmatrix} a & 0 \\ 0 & 1/a \end{pmatrix} : a \in \mathbb{R}^\times\}$$

and [5]

$$P = NM = MN$$

The **unramified principal series** representation [6] I_s is as a space of smooth [7] functions f on G with the prescribed left equivariance

$$I_s = \{f : f(nmg) = \chi_s(p)\, f(g) \text{ for all } p \in P,\ g \in G\}$$

where $s \in \mathbb{C}$ and

$$\chi_s \begin{pmatrix} a & * \\ 0 & a^{-1} \end{pmatrix} = |a|^{2s}$$

The group G acts on I_s by the **right regular** representation, that is, by right translation of functions: [8]

$$(g \cdot f)(x) = f(xg)$$

for $g, x \in G$. This action preserves smoothness, and since the action is *right* translation it does not disturb the defining property of *left* equivariance by P.

The **standard intertwining operator** [9] $T = T_s : I_s \to I_{1-s}$ is defined, for $\mathrm{Re}(s)$ sufficiently large, by

[4] The usual terminology would refer to P as the *standard parabolic*, M its *standard Levi component*, and N its *unipotent radical*. One need not know what sense these terms might have in general to follow the present discussion. Rather, the simple objects to which the terms refer here might suggest that even the general sense of the terms might be not too mysterious.

[5] *Why this subgroup?* The subgroup P arises naturally in many ways, among which is the fact that the quotient $P\backslash G$ is *compact*. In this sense, P is a *large* subgroup of G. Further, it is a semi-direct product of N and M, the former isomorphic to the real line with addition, the latter isomorphic to the multiplicative group of real numbers. Thus, P is composed of parts already familiar to us.

[6] There is no *series* here in the mathematical sense, but, rather, a *continuum* of representations parametrized by the complex variable s. They are *principal* in the colloquial sense of constituting a large chunk of the totality of all (more-or-less irreducible) representations of G. They are *unramified* in the sense that the relevant homomorphisms $\mathbb{R}^\times \to \mathbb{C}^\times$ factor through the absolute value on $\mathbb{R}^\times$. The dependence of the unramified principal series on the parameter s is often normalized, with the benefit of hindsight, in a different form than the innocent form we use here. Specifically, our s would often be $s+1$, or $2s+1$.

[7] Here *smooth* has the usual sense of *infinitely differentiable.*

[8] *Why this space of functions?* There are at least two parts to this question. *Induction* from representations of a subgroup can be viewed as an *adjoint functor* to the forgetful functor of *restriction* of a representation to the subgroup. That is, induction is a very reasonable way to make representations. Further, spaces of functions on a group G itself are convenient *models* of isomorphism classes of representations.

[9] This terminology is inherited from physics. In any case, an intertwining operator is simply a morphism of representation spaces for a fixed group.

the integral [10]

$$(T_s f)(g) = \int_N f(wn \cdot g)\, dn$$

where the longest Weyl element [11] w is

$$w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Convergence of this integral will be clarified shortly. Again, since the map is defined as an integration on the left, it does not disturb the right action of G. To verify that (assuming convergence) the image really does lie inside I_{1-s}, observe that $T_s f$ is left N-invariant by construction, and that for $m \in M$

$$(T_s f)(mg) = \int_N f(wn \cdot mg)\, dn = \int_N f(wm\, m^{-1}nm \cdot g)\, dn = \chi_1(m) \cdot \int_N f(wm\, n \cdot g)\, dn$$

by replacing n by mnm^{-1}, taking into account the change of measure $d(mnm^{-1}) = \chi_1(m) \cdot dn$ coming from

$$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}^{-1} = \begin{pmatrix} 1 & a^2 x \\ 0 & 1 \end{pmatrix}$$

Then this is

$$\chi_1(m) \cdot \int_N f(wmw^{-1} \cdot w\, n \cdot g)\, dn = \chi_1(m) \cdot \int_N f(m^{-1} \cdot w\, n \cdot g)\, dn$$

$$= \chi_1(m)\chi_s(m^{-1}) \cdot \int_N f(w\, n \cdot g)\, dn = \chi_{1-s}(m) \cdot (T_s f)(g)$$

This verifies that $T_s : I_s \to I_{1-s}$.

Recall [12] the **Iwasawa decomposition**

$$G = P \cdot K$$

where [13]

$$K = \{ \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} : \theta \in \mathbb{R}/2\pi i\mathbb{Z} \}$$

and note that the overlap is just $P \cap K = \pm 1$. Thus, a function f in I_s is completely determined by its values on K, in fact, on $\{\pm 1\}\backslash K$. Conversely, for fixed $s \in \mathbb{C}$, any smooth function f_o on $\{\pm 1\}\backslash K$ has a unique extension (depending upon s) to a function $f \in I_s$, by

$$f(pk) = \chi_s(p) \cdot f_o(k)$$

[10] *Why this integral?* This is an analogue of a finite-group method for writing formulas for intertwining operators from a representation induced from a subgroup A to a representation induced from a subgroup B, with intertwining operators roughly corresponding to double cosets $A\backslash G/B$. For finite groups, this goes by the name of *Mackey theory*, and Bruhat extended the idea to Lie groups and p-adic groups. For non-finite groups, there are issues of convergence and analytic continuation. In any case, without further justification here, this integral is one that arises naturally.

[11] The 2-by-2 matrix at hand could certainly be mentioned without using this terminology, but it is constructive to introduce general terminology in simple examples. Note that the Bruhat decomposition is $G = P \cup PwP$. That is, $P\backslash G/P$ has cardinality 2, and Mackey-Bruhat theory would tell us that there is just one possibility for an intertwining operator, namely the one given by this integral.

[12] If necessary, from a former life? Anyway, we'll only use an explicit form, so no general principle is necessary. Instead, again, it is constructive to introduce general terminology in simple examples.

[13] This subgroup of G is a *special orthogonal group*, denoted $SO(2)$, but we need nothing from this family of facts. It is also a *maximal* compact subgroup of G, but, we do not need this, either.

Taking advantage of the simplicity[14] of this situation, we may expand smooth functions on K in classical **Fourier series**

$$f\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \sum_{n\in\mathbb{Z}} c_n\, e^{2\pi i n\theta}$$

where the Fourier coefficients c_n are *rapidly decreasing* due to the smoothness of f.

In studying the intertwining operators T_s, it turns out to be wise to restrict our attention to functions $f \in I_s$ which are not merely smooth, but in fact are **right K-finite** in the sense that the Fourier expansion of f restricted to K is *finite*. Thus, these functions will be finite sums of very simple functions in I_s of the form

$$f(pk) = \chi_s(p)\cdot\rho_n(k)$$

where

$$\rho_n\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \left(e^{i\theta}\right)^n$$

For any function f on G with

$$f(pk) = f(g)\cdot\rho(k) \quad (\text{for } k\in K)$$

with ρ among the ρ_n, say that f has (right) **K-type** ρ. If we believe the Iwasawa decomposition $G = PK$, then

$$\dim_{\mathbb{C}}\{f\in I_s :\ f \text{ has right } K\text{-type } \rho_n\} = \begin{cases} 1 & (n \text{ even}) \\ 0 & (n \text{ odd}) \end{cases}$$

In other words, the *multiplicities*[15] of the K-types in I_s are just 1 and 0.

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2. *The main computation*

We will directly compute the effect of the intertwining operator T_s on a function f in I_s with a fixed right K-type ρ. Since the left integration over N cannot affect the right K-type, T_s preserves K-types. Since the dimensions of the subspaces of I_s and I_{1-s} with given K-type ρ are 1 (or 0), necessarily T_s maps the function

$$f(pk) = \chi_s(p)\cdot\rho(k)$$

to some *multiple* of the function

$$\varphi(pk) = \chi_{1-s}(p)\cdot\rho(k)$$

To determine this constant, it suffices to evaluate $(T_sf)(1)$, that is, to evaluate the integral

$$(T_sf)(1) = \int_N f(wn)\,dn = \int_{\mathbb{R}} f(wn_x)\,dx \qquad (\text{with } n_x = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix})$$

To evaluate $f(wn_x)$, we must give the Iwasawa decomposition $wn_x = pk$. One convenient approach is to compute

$$(wn_x)(wn_x)^\top = (pk)(pk)^\top = pkk^{-1}p^\top = pp^\top$$

since k is orthogonal. Letting

$$p = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}$$

[14] In more than two dimensions special orthogonal groups $SO(n)$ are not abelian.

[15] In a conflicting but equally common use of the word *multiplicity*, we would also say that as a representation space for K the space I_s is **multiplicity-free**.

and expanding $(wn_x)(wn_x)^\top$ gives

$$\begin{pmatrix} 1 & -x \\ -x & 1+x^2 \end{pmatrix} = \begin{pmatrix} a^2+b^2 & b/a \\ b/a & 1/a^2 \end{pmatrix}$$

from which $a^{-2} = 1+x^2$ and $b/a = -x$, so $a = 1/\sqrt{1+x^2}$ and $b = -x/\sqrt{1+x^2}$. Then $k = p^{-1}g$, so we find the Iwasawa decomposition

$$wn_x = \begin{pmatrix} \frac{1}{\sqrt{1+x^2}} & \frac{-x}{\sqrt{1+x^2}} \\ 0 & \sqrt{1+x^2} \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\sqrt{1+x^2}} & \frac{x}{\sqrt{1+x^2}} \\ \frac{-x}{\sqrt{1+x^2}} & \frac{1}{\sqrt{1+x^2}} \end{pmatrix}$$

Thus, with K-type $\rho = \rho_{2n}$, with $2n \in 2\mathbb{Z}$,

$$f(wn_x) = f(pk) = \chi_s(p)\cdot\rho_{2n}(k) = (\frac{1}{\sqrt{1+x^2}})^{2s}\cdot(\frac{1+ix}{\sqrt{1+x^2}})^{2n} = (1+x^2)^{-s}\cdot(\frac{1+ix}{\sqrt{1+ix}\cdot\sqrt{1-ix}})^{2n}$$

$$= (1+x^2)^{-s}\cdot(\frac{1+ix}{1-ix})^n = (1+ix)^{-s+n}\,(1-ix)^{-s-n}$$

Thus, our intertwining operator when applied to a vector [16] $f \in I_s$ with specified K-type ρ_{2n}, evaluated at $1 \in G$ is

$$(T_s f)(1) = \int_N f(wn)\,dn = \int_{\mathbb{R}} (1+ix)^{-s+n}\,(1-ix)^{-s-n}\,dx$$

To compute the latter, we use a standard trick employing the gamma function. That is, for complex z in the right half-plane, and for $\mathrm{Re}(s) > 0$,

$$\Gamma(s)\cdot z^{-s} = \int_0^\infty e^{-tz}t^{-s}\,\frac{dt}{t}$$

Thus,

$$(T_s f)(1) = \int_{\mathbb{R}} (1+ix)^{-s+n}\,(1-ix)^{-s-n}\,dx$$

$$= \Gamma(s-n)^{-1}\Gamma(s+n)^{-1}\cdot\int_{\mathbb{R}}\int_0^\infty\int_0^\infty e^{-u(1+ix)}u^{-(s-n)}\,e^{-v(1+ix)}v^{-(s+n)}\,\frac{du}{u}\,\frac{dv}{v}\,dx$$

Changing the order of integration and integrating in x first [17] gives an inner integral

$$\int_{\mathbb{R}} e^{ix(u-v)}\,dx = 2\pi\cdot\delta_{u-v}$$

where δ is the Dirac delta distribution. [18] Thus, the whole integral becomes

$$(T_s f)(1) = \frac{2\pi}{\Gamma(s-n)\Gamma(s+n)}\int_0^\infty e^{-u}u^{-(s-n)}\,e^{-u}u^{-(s+n)}u^{-1}u^{-1}du$$

$$= \frac{2\pi}{\Gamma(s-n)\Gamma(s+n)}\int_0^\infty e^{-2u}u^{-2s-1}\,\frac{du}{u} = \frac{2\pi\,2^{1-2s}\,\Gamma(2s-1)}{\Gamma(s-n)\Gamma(s+n)}$$

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[16] The space of these functions is a vector space, certainly.

[17] This is not legitimate from an elementary viewpoint. However, it is a compelling heuristic, correctly suggests the true conclusion, and can immediately be justified by **Fourier inversion**, as is done in the appendix where the gamma function is discussed.

[18] From this viewpoint it is a little difficult to account for the factor of 2π, but this becomes understandable when the computation is redone via Fourier inversion. In any case, the constant is irrelevant to vanishing and non-vanishing.

That is, under the intertwining $T_s : I_s \to I_{1-s}$, the function f with right K-type ρ_{2n} normalized such that $f(1) = 1$ is mapped to the similarly-normalized function in I_{1-s} with the same K-type, multiplied by that last constant.

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3. *Subrepresentations*

For brevity, let

$$\lambda(s,n) = \frac{2\pi\, 2^{1-2s}\, \Gamma(2s-1)}{\Gamma(s-n)\Gamma(s+n)}$$

denote the constant computed above. The intertwining operator T_s is holomorphic [19] at $s_o \in \mathbb{C}$ if for all integers n the function $\lambda(s,n)$ is holomorphic at s_o.

The numerator $\Gamma(2s-1)$ has poles at

$$\frac{1}{2},\ 0,\ -\frac{1}{2},\ -1,\ -\frac{3}{2},\ -2,\ \ldots$$

The half-integer poles are not canceled by the poles of the denominator, so T_s has poles at these half-integers. At the non-positive integers, regardless of the value of n the poles of the denominator cancel the pole of the numerator. That is, [20]

[3.0.1] **Proposition:** $T_s : I_s \to I_{1-s}$ is holomorphic away from

$$s = \frac{1}{2},\ -\frac{1}{2},\ -\frac{3}{2},\ -\frac{5}{2},\ -\frac{7}{2},\ \ldots$$

at which it has simple poles. ///

For s *not* an integer, the denominator has no poles, so (away from the half-integers at which the numerator has a pole) $\lambda(s,n) \neq 0$ for *all* K-types ρ_{2n}. Thus,

[3.0.2] **Proposition:** The intertwining operator $T_s : I_s \to I_{1-s}$ has trivial kernel for s not an integer (and away from its poles). ///

Consider $s = \ell$ with $0 < \ell \in \mathbb{Z}$. The numerator has no pole at ℓ, while the denominator has a pole, yielding a $\lambda(\ell, n) = 0$ for all integers

$$n = \pm\ell,\ \pm(\ell+1),\ \pm(\ell+2),\ \pm(\ell+3),\ \ldots$$

Thus, for $0 < \ell \in \mathbb{Z}$, I_ℓ has a non-trivial infinite-dimensional subrepresentation [21] consisting of these K-types in the kernel of $T_\ell : I_\ell \to I_{1-\ell}$.

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[19] One should be wary of the notion of *holomorphy* of an intertwining-operator-valued function of a complex variable. While many standard sources do treat holomorphic Hilbert-space-valued and Banach-space-valued functions, or even Fréchet-space-valued functions, greater generality is rare. Nevertheless, it does turn out that our usual expectations of Cauchy theory are fulfilled for a very broad class of topological vector spaces, namely *locally convex, quasi-complete* spaces. The notion of local convexity is standard, but quasi-completeness is less so. But, in fact, essentially all the topological vector spaces of interest fall into this class, including spaces of distributions, spaces of operators with weak topologies, etc. Thus, mildly ironically, everything does turn out just fine, whether or not one worried about it.

[20] Recall that $\Gamma(s)$ has *no zeros*, and simple poles at non-positive integers.

[21] These subrepresentations have names, based on how they arose in other circumstances: are the sum of the **holomorphic discrete series** and **anti-holomorphic discrete series** representations.

Consider $s = -\ell$ with $0 \geq -\ell \in \mathbb{Z}$. The numerator has a pole at $-\ell$, and the denominator has a *double* pole for integers

$$n = 0,\ \pm 1,\ \pm 2,\ \ldots,\ \pm \ell$$

and a *single* pole for integers

$$n = \pm(\ell+1),\ \pm(\ell+2),\ \pm(\ell+3),\ \ldots$$

Thus, $\lambda(-\ell, n) = 0$ for the double poles, and the single poles cancel. Thus, for $0 \geq \ell \in \mathbb{Z}$, I_ℓ has a non-trivial subrepresentation consisting of the finitely-many K-types at which the denominator has a double pole. These are (therefore) *finite-dimensional* representations, the kernels of $T_{-\ell} : I_{-\ell} \to I_{1+\ell}$.

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4. *Return to smooth vectors*

The explicit computation of the scalar $\lambda(s, 2n) = (T_s f_{2n})(1)$ for f_{2n} the normalized vector in I_s with K-type ρ_{2n} also shows that T_s has an analytic continuation on *smooth* vectors in I_s, nor merely K-finite vectors, as follows. From $\Gamma(s) \cdot s = \Gamma(s+1)$ we see that

$$(T_s f_{2n})(1) = \frac{2\pi\, 2^{1-2s}\, \Gamma(2s-1)}{\Gamma(s-n)\Gamma(s+n)} = \text{polynomial growth in } n$$

Then let

$$f = \sum_{n \in \mathbb{Z}} c_{2n} \cdot f_{2n}$$

be smooth in I_s. Smoothness is equivalent to the **rapid decrease** [22] of the Fourier coefficients. Then

$$T_s f = \sum_{n \in \mathbb{Z}} \lambda(s, 2n) \cdot c_{2n} \cdot f_{2n}$$

still has rapidly decreasing coefficients, so is a smooth vector in I_{1-s}. That is, away from the poles, the intertwining operator T_s when analytically continued *is* defined on all smooth vectors in I_{1-s}, not merely K-finite ones.

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5. *Appendix: usual tricks with* $\Gamma(s)$

The property of $\Gamma(s)$ used above is standard, but sufficiently important that we review it. The gamma function is given for $\mathrm{Re}(s) > 0$ by Euler's integral

$$\Gamma(s) = \int_0^\infty e^{-t} t^s \, \frac{dt}{t}$$

Replacing t by ty with $y > 0$

$$\Gamma(s) \cdot y^{-s} = \int_0^\infty e^{-ty} t^s \, \frac{dt}{t}$$

By analytic continuation to the right complex half-plane, for $y > 0$ and $x \in \mathbb{R}$

$$\Gamma(s) \cdot (y + 2\pi i x)^{-s} = \int_0^\infty e^{-t(y+2\pi i x)} t^s \, \frac{dt}{t}$$

[22] *Rapid decrease* means, as usual, that $|n|^N \cdot |c_{2n}|$ goes to 0 (as $|n| \to \infty$) for every N. This equivalence is readily proven by repeated integration by parts in the integrals yielding the Fourier coefficients.

Having analytically continued, we may let $y = 1$ again, obtaining

$$\Gamma(s) \cdot (1 + 2\pi i x)^{-s} = \int_0^\infty e^{-t(1+2\pi i x)} t^s \, \frac{dt}{t} = \int_0^\infty e^{-2\pi i x t} \, e^{-t} t^s \, \frac{dt}{t}$$

which is the Fourier transform of

$$\varphi_s(t) = \begin{cases} e^{-t} \, t^{s-1} & (t > 0) \\ 0 & (t < 0) \end{cases}$$

To compute the concrete integral for $(T_s f)(1)$ we invoke the Plancherel theorem, that

$$\int_{\mathbb{R}} f(x) \, \overline{\varphi(x)} \, dx = \int_{\mathbb{R}} \hat{f}(x) \, \overline{\hat{\varphi}(x)} \, dx$$

and Fourier inversion. Then, with real $s \gg 0$, replacing x by $2\pi x$ at the first step, and with real s,

$$\int_{\mathbb{R}} (1 + ix)^{-s+n} \, (1 - ix)^{-s-n} \, dx = 2\pi \int_{\mathbb{R}} (1 + 2\pi i x)^{-s+n} \, (1 - 2\pi i x)^{-s-n} \, dx$$

$$= 2\pi \int_{\mathbb{R}} \hat{\varphi}_{s-n}(x) \, \overline{\hat{\varphi}_{s+n}(x)} \, dx = 2\pi \int_{\mathbb{R}} \varphi_{s-n}(x) \, \overline{\varphi_{s+n}(x)} \, dx$$

$$= \frac{2\pi}{\Gamma(s-n)\,\Gamma(s+n)} \int_0^\infty e^{-u} u^{-(s-n)-1} \cdot e^{-u} u^{-(s+n)-1} \, du = \frac{2\pi \Gamma(2s-1)}{\Gamma(s-n)\,\Gamma(s+n)}$$

as computed more heuristically earlier. This also exhibits the constant 2π.
