Let $G = SL(2, \mathbb{R})$. Our goal is to find irreducible representations of $G$. The actual work we do is quite prosaic, and we do find several sorts of representations of $G$, without any fancier ideas.

However, the completeness of our computation depends on some more serious results. Further, the motivation for this computation, and the idea that it would be fruitful, come from less pedestrian thinking.

The computation itself can be understood on its own terms without understanding the broader context. For that matter, the fact that a few simple if non-elementary ideas lead to such tangible computational results might be construed as a motivation to study those non-elementary notions.

To understand the larger context, first note that Casselman’s subrepresentation theorem asserts that any irreducible representation $\pi$ of $G$ is a subrepresentation of some one of the principal series representations $I_s$ defined below. Further, the quotient $I_s/\pi$ (or a further quotient that is irreducible) again imbeds in some $I_{s'}$ with another parameter value $s'$. Thus, any irreducible $\pi$ appears as a (possibly subrepresentation of a) kernel of a $G$-homomorphism $I_s \to I_{s'}$ among principal series. Still further, examination of the eigenvalues of the center of the enveloping algebra on adroitly chosen vectors in the principal series. But, again, the computation itself is understandable without necessarily fully appreciating this grounding of it.

This computation also gives an approach to understand why the gamma function should not vanish.

- Principal series representations
- The main computation
- Subrepresentations
- Return to smooth vectors
- Appendix: usual tricks with $\Gamma(s)$

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[1] To be more accurate, the subrepresentation theorem in fact asserts the imbeddability of irreducible $(\mathfrak{g}, K)$ representations, also called $(\mathfrak{g}, K)$-modules. The aptness of this notion was one of Harish-Chandra’s basic and indispensable contributions to this subject. Here $\mathfrak{g}$ is the Lie algebra of $G$ acting by the differentiated version of the action of $G$ (at least on smooth vectors of the representation), and $K$ is a maximal compact subgroup of $G$. Thus, the $(\mathfrak{g}, K)$-module structure forgets some of the structure of a $G$-representation.

[2] Actually, our present discussion only discusses half the principal series, namely the even or unnamified ones.

[3] The enveloping algebra is the associative algebra generated by $\mathfrak{g}$ with relations $xy - yx = [x, y]$ for $x, y \in \mathfrak{g}$. The structure of its center is described by an early theorem of Harish-Chandra, and by a Schur lemma the center acts by scalars on irreducibles. Further, it is our good fortune that the eigenvalues of the center quite successfully distinguish among principal series.
1. **Principal series representations**

Define useful subgroups\(^4\) of \(G\) by

\[
N = \{ n_x = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{R} \} \quad M = \{ \begin{pmatrix} a & 0 \\ 0 & 1/a \end{pmatrix} : a \in \mathbb{R}^\times \} \]

and\(^5\)

\[ P = NM = MN \]

The **unramified principal series** representation\(^6\) \(I_s\) is as a space of smooth\(^7\) functions \(f\) on \(G\) with the prescribed left equivariance

\[ I_s = \{ f : f(nmg) = \chi_s(p) f(g) \text{ for all } p \in P, g \in G \} \]

where \(s \in \mathbb{C}\) and

\[ \chi_s \left( \begin{array}{cc} a & * \\ 0 & a^{-1} \end{array} \right) = |a|^{2s} \]

The group \(G\) acts on \(I_s\) by the **right regular** representation, that is, by right translation of functions:\(^8\)

\[ (g \cdot f)(x) = f(xg) \]

for \(g, x \in G\). This action preserves smoothness, and since the action is right translation it does not disturb the defining property of left equivariance by \(P\).

The **standard intertwining operator**\(^9\) \(T = T_s : I_s \to I_{1-s}\) is defined, for \(\text{Re}(s)\) sufficiently large, by

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\(^4\) The usual terminology would refer to \(P\) as the **standard parabolic**, \(M\) its **standard Levi component**, and \(N\) its **unipotent radical**. One need not know what sense these terms might have in general to follow the present discussion. Rather, the simple objects to which the terms refer here might suggest that even the general sense of the terms might be not too mysterious.

\(^5\) **Why this subgroup?** The subgroup \(P\) arises naturally in many ways, among which is the fact that the quotient \(P \backslash G\) is **compact**. In this sense, \(P\) is a large subgroup of \(G\). Further, it is a semi-direct product of \(N\) and \(M\), the former isomorphic to the real line with addition, the latter isomorphic to the multiplicative group of real numbers. Thus, \(P\) is composed of parts already familiar to us.

\(^6\) **There is no series here in the mathematical sense, but, rather, a continuum of representations parametrized by the complex variable \(s\). They are principal in the colloquial sense of constituting a large chunk of the totality of all (more-or-less irreducible) representations of \(G\). They are unramified in the sense that the relevant homomorphisms \(\mathbb{R}^\times \to \mathbb{C}^\times\) factor through the absolute value on \(\mathbb{R}^\times\). The dependence of the unramified principal series on the parameter \(s\) is often normalized, with the benefit of hindsight, in a different form than the innocent form we use here. Specifically, our \(s\) would often be \(s + 1\), or \(2s + 1\).**

\(^7\) **Here smooth has the usual sense of infinitely differentiable.**

\(^8\) **Why this space of functions?** There are at least two parts to this question. **Induction** from representations of a subgroup can be viewed as an adjoint functor to the forgetful functor of restriction of a representation to the subgroup. That is, induction is a very reasonable way to make representations. Further, spaces of functions on a group \(G\) itself are convenient models of isomorphism classes of representations.

\(^9\) This terminology is inherited from physics. In any case, an intertwining operator is simply a morphism of representation spaces for a fixed group.
The 2-by-2 matrix at hand could certainly be mentioned without using this terminology, but it is constructive to introduce general terminology in simple examples. Instead, again, it is constructive to introduce general terminology in simple examples.

Convergence of this integral will be clarified shortly. Again, since the map is defined as an integration on $G/B$, from a former life? Anyway, we’ll only use an explicit form, so no general principle is necessary.

Recall the Iwasawa decomposition

$$G = P \cdot K$$

where

$$K = \left\{ \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} : \theta \in \mathbb{R}/2\pi \mathbb{Z} \right\}$$

and note that the overlap is just $P \cap K = \pm 1$. Thus, a function $f$ in $I_s$ is completely determined by its values on $K$, in fact, on $\{ \pm 1 \} \setminus K$. Conversely, for fixed $s \in \mathbb{C}$, any smooth function $f_o$ on $\{ \pm 1 \} \setminus K$ has a unique extension (depending upon $s$) to a function $f \in I_s$, by

$$f(pk) = \chi_s(p) \cdot f_o(k)$$

Why this integral? This is an analogue of a finite-group method for writing formulas for intertwining operators from a representation induced from a subgroup $A$ to a representation induced from a subgroup $B$, with intertwining operators roughly corresponding to double cosets $A \backslash G/B$. For finite groups, this goes by the name of Mackey theory, and Bruhat extended the idea to Lie groups and $p$-adic groups. For non-finite groups, there are issues of convergence and analytic continuation. In any case, without further justification here, this integral is one that arises naturally.

The 2-by-2 matrix at hand could certainly be mentioned without using this terminology, but it is constructive to introduce general terminology in simple examples. Note that the Bruhat decomposition is $G = P \cup PwP$. That is, $P \backslash G/P$ has cardinality 2, and Mackey-Bruhat theory would tell us that there is just one possibility for an intertwining operator, namely the one given by this integral.

If necessary, from a former life? Anyway, we’ll only use an explicit form, so no general principle is necessary. Instead, again, it is constructive to introduce general terminology in simple examples.

This subgroup of $G$ is a special orthogonal group, denoted $SO(2)$, but we need nothing from this family of facts. It is also a maximal compact subgroup of $G$, but, we do not need this, either.
Taking advantage of the simplicity\[14\] of this situation, we may expand smooth functions on $K$ in classical Fourier series

$$f\left(\begin{array}{cc}
cos\theta & \sin\theta \\
-sin\theta & cos\theta
\end{array}\right) = \sum_{n \in \mathbb{Z}} c_n e^{2\pi in\theta}$$

where the Fourier coefficients $c_n$ are rapidly decreasing due to the smoothness of $f$.

In studying the intertwining operators $T_s$, it turns out to be wise to restrict our attention to functions $f \in I_s$ which are not merely smooth, but in fact are right $K$-finite in the sense that the Fourier expansion of $f$ restricted to $K$ is finite. Thus, these functions will be finite sums of very simple functions in $I_s$ of the form

$$f(pk) = \chi_s(p) \cdot \rho_n(k)$$

where

$$\rho_n\left(\begin{array}{cc}
cos\theta & \sin\theta \\
-sin\theta & cos\theta
\end{array}\right) = (e^{i\theta})^n$$

For any function $f$ on $G$ with

$$f(pk) = f(g) \cdot \rho(k) \quad (\text{for } k \in K)$$

with $\rho$ among the $\rho_n$, say that $f$ has (right) $K$-type $\rho$. If we believe the Iwasawa decomposition $G = PK$, then

$$\dim_{\mathbb{C}} \{ f \in I_s : f \text{ has right } K\text{-type } \rho_n \} = \begin{cases} 1 & (n \text{ even}) \\ 0 & (n \text{ odd}) \end{cases}$$

In other words, the multiplicities\[15\] of the $K$-types in $I_s$ are just 1 and 0.

### 2. The main computation

We will directly compute the effect of the intertwining operator $T_s$ on a function $f$ in $I_s$ with a fixed right $K$-type $\rho$. Since the left integration over $N$ cannot affect the right $K$-type, $T_s$ preserves $K$-types. Since the dimensions of the subspaces of $I_s$ and $I_{1-s}$ with given $K$-type $\rho$ are 1 (or 0), necessarily $T_s$ maps the function

$$f(pk) = \chi_s(p) \cdot \rho(k)$$

to some multiple of the function

$$\varphi(pk) = \chi_{1-s}(p) \cdot \rho(k)$$

To determine this constant, it suffices to evaluate $(T_s f)(1)$, that is, to evaluate the integral

$$(T_s f)(1) = \int_N f(wn) \, dn = \int_{\mathbb{R}} f(wn_x) \, dx \quad (\text{with } n_x = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix})$$

To evaluate $f(wn_x)$, we must give the Iwasawa decomposition $wn_x = pk$. One convenient approach is to compute

$$(wn_x)(wn_x)^\top = (pk)(pk)^\top = pkk^{-1}p^\top = pp^\top$$

since $k$ is orthogonal. Letting

$$p = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}$$

\[14\] In more than two dimensions special orthogonal groups $SO(n)$ are not abelian.

\[15\] In a conflicting but equally common use of the word multiplicity, we would also say that as a representation space for $K$ the space $I_s$ is multiplicity-free.
and expanding $\langle wn_x, wn_x \rangle^T$ gives
\[
\begin{pmatrix}
1 & -x \\
-x & 1 + x^2
\end{pmatrix}
= \begin{pmatrix}
a^2 + b^2 & b/a \\
b/a & 1/a^2
\end{pmatrix}
\]
from which $a^{-2} = 1 + x^2$ and $b/a = -x$, so $a = 1/\sqrt{1 + x^2}$ and $b = -x/\sqrt{1 + x^2}$. Then $k = p^{-1}g$, so we find the Iwasawa decomposition
\[
wn_x = \begin{pmatrix}
\frac{1}{\sqrt{1 + x^2}} & \frac{-x}{\sqrt{1 + x^2}} \\
0 & \frac{1}{\sqrt{1 + x^2}}
\end{pmatrix}
\begin{pmatrix}
\frac{x}{\sqrt{1 + x^2}} \\
\frac{-x}{\sqrt{1 + x^2}}
\end{pmatrix}
\]
Thus, with $K$-type $\rho = \rho_{2n}$, with $2n \in \mathbb{Z}$,
\[
f(wn_x) = f(pk) = \chi_s(p) \cdot \rho_{2n}(k) = \left(\frac{1}{\sqrt{1 + x^2}}\right)^{2s} \cdot \left(\frac{1 + ix}{\sqrt{1 + x^2}}\right)^{2n} = (1 + x^2)^{-s} \cdot \left(\frac{1 + ix}{\sqrt{1 + ix}}\cdot \sqrt{1 - ix}\right)^{2n}
\]
\[
= (1 + x^2)^{-s} \cdot \left(\frac{1 + ix}{1 - ix}\right)^n = (1 + ix)^{-s+n} (1 - ix)^{-s-n}
\]
Thus, our intertwining operator when applied to a vector, with specified $K$-type $\rho_{2n}$, evaluated at $1 \in G$ is
\[
(T_s f)(1) = \int_N f(wn) \, d\mu = \int_{\mathbb{R}} (1 + ix)^{-s+n} (1 - ix)^{-s-n} \, dx
\]
To compute the latter, we use a standard trick employing the gamma function. That is, for complex $z$ in the right half-plane, and for $\Re(s) > 0$,
\[
\Gamma(s) \cdot z^{-s} = \int_0^{\infty} e^{-tz} t^{-s} \, dt
\]
Thus,
\[
(T_s f)(1) = \int_{\mathbb{R}} (1 + ix)^{-s+n} (1 - ix)^{-s-n} \, dx
\]
\[
= \Gamma(s - n)^{-1} \Gamma(s + n)^{-1} \cdot \int_0^{\infty} \int_0^{\infty} e^{-u(1+ix)} u^{-(s-n)} e^{-v(1+ix)} v^{-(s+n)} \, \frac{du}{u} \, \frac{dv}{v} \, dx
\]
Changing the order of integration and integrating in $x$ first gives an inner integral
\[
\int_{\mathbb{R}} e^{ix(u-v)} \, dx = 2\pi \cdot \delta_{u-v}
\]
where $\delta$ is the Dirac delta distribution. Thus, the whole integral becomes
\[
(T_s f)(1) = \frac{2\pi}{\Gamma(s-n) \Gamma(s+n)} \int_0^{\infty} e^{-u(-s-n)} e^{-u(s+n)} u^{-1} \, du
\]
\[
\quad = \frac{2\pi}{\Gamma(s-n) \Gamma(s+n)} \int_0^{\infty} e^{-2u} \, du \quad = \frac{2\pi \, 2^{1-2s} \Gamma(2s-1)}{\Gamma(s-n) \Gamma(s+n)}
\]
[16] The space of these functions is a vector space, certainly.

[17] This is not legitimate from an elementary viewpoint. However, it is a compelling heuristic, correctly suggests the true conclusion, and can immediately be justified by Fourier inversion, as is done in the appendix where the gamma function is discussed.

[18] From this viewpoint it is a little difficult to account for the factor of $2\pi$, but this becomes understandable when the computation is redone via Fourier inversion. In any case, the constant is irrelevant to vanishing and non-vanishing.
That is, under the intertwining $T_s : I_s \to I_{1-s}$, the function $f$ with right $K$-type $\rho_{2n}$ normalized such that $f(1) = 1$ is mapped to the similarly-normalized function in $I_{1-s}$ with the same $K$-type, multiplied by that last constant.

### 3. Subrepresentations

For brevity, let

$$\lambda(s, n) = \frac{2\pi 2^{1-2s} \Gamma(2s-1)}{\Gamma(s-n)\Gamma(s+n)}$$

denote the constant computed above. The intertwining operator $T_s$ is holomorphic\footnote{One should be wary of the notion of holomorphy of an intertwining-operator-valued function of a complex variable. While many standard sources do treat holomorphic Hilbert-space-valued and Banach-space-valued functions, or even Fréchet-space-valued functions, greater generality is rare. Nevertheless, it does turn out that our usual expectations of Cauchy theory are fulfilled for a very broad class of topological vector spaces, namely locally convex, quasi-complete spaces. The notion of local convexity is standard, but quasi-completeness is less so. But, in fact, essentially all the topological vector spaces of interest fall into this class, including spaces of distributions, spaces of operators with weak topologies, etc. Thus, mildly ironically, everything does turn out just fine, whether or not one worried about it.} at $s_o \in \mathbb{C}$ if for all integers $n$ the function $\lambda(s, n)$ is holomorphic at $s_o$.

The numerator $\Gamma(2s-1)$ has poles at

$$\frac{1}{2}, 0, -\frac{1}{2}, -1, -\frac{3}{2}, -2, \ldots$$

The half-integer poles are not canceled by the poles of the denominator, so $T_s$ has poles at these half-integers. At the non-positive integers, regardless of the value of $n$ the poles of the denominator cancel the pole of the numerator. That is,\footnote{Recall that $\Gamma(s)$ has no zeros, and simple poles at non-positive integers.}

\[3.0.1\] **Proposition:** $T_s : I_s \to I_{1-s}$ is holomorphic away from

$$s = \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, -\frac{7}{2}, \ldots$$

at which it has simple poles. //\

For $s$ not an integer, the denominator has no poles, so (away from the half-integers at which the numerator has a pole) $\lambda(s, n) \neq 0$ for all $K$-types $\rho_{2n}$. Thus,

\[3.0.2\] **Proposition:** The intertwining operator $T_s : I_s \to I_{1-s}$ has trivial kernel for $s$ not an integer (and away from its poles). //\

Consider $s = \ell$ with $0 < \ell \in \mathbb{Z}$. The numerator has no pole at $\ell$, while the denominator has a pole, yielding a $\lambda(\ell, n) = 0$ for all integers

$$n = \pm \ell, \pm(\ell + 1), \pm(\ell + 2), \pm(\ell + 3), \ldots$$

Thus, for $0 < \ell \in \mathbb{Z}$, $I_{\ell}$ has a non-trivial infinite-dimensional subrepresentation\footnote{These subrepresentations have names, based on how they arose in other circumstances: are the sum of the holomorphic discrete series and anti-holomorphic discrete series representations.} consisting of these $K$-types in the kernel of $T_{\ell} : I_{\ell} \to I_{1-\ell}$.

\[19\] One should be wary of the notion of holomorphy of an intertwining-operator-valued function of a complex variable. While many standard sources do treat holomorphic Hilbert-space-valued and Banach-space-valued functions, or even Fréchet-space-valued functions, greater generality is rare. Nevertheless, it does turn out that our usual expectations of Cauchy theory are fulfilled for a very broad class of topological vector spaces, namely locally convex, quasi-complete spaces. The notion of local convexity is standard, but quasi-completeness is less so. But, in fact, essentially all the topological vector spaces of interest fall into this class, including spaces of distributions, spaces of operators with weak topologies, etc. Thus, mildly ironically, everything does turn out just fine, whether or not one worried about it.

\[20\] Recall that $\Gamma(s)$ has no zeros, and simple poles at non-positive integers.
Consider $s = -\ell$ with $0 \geq -\ell \in \mathbb{Z}$. The numerator has a pole at $-\ell$, and the denominator has a double pole for integers
\[ n = 0, \pm 1, \pm 2, \ldots, \pm \ell \]
and a single pole for integers
\[ n = \pm (\ell + 1), \pm (\ell + 2), \pm (\ell + 3), \ldots \]
Thus, $\lambda(-\ell, n) = 0$ for the double poles, and the single poles cancel. Thus, for $0 \geq \ell \in \mathbb{Z}$, $I_{-\ell}$ has a non-trivial subrepresentation consisting of the finitely-many $K$-types at which the denominator has a double pole. These are (therefore) finite-dimensional representations, the kernels of $T_{-\ell} : I_{-\ell} \rightarrow I_{1+\ell}$.

4. **Return to smooth vectors**

The explicit computation of the scalar $\lambda(s, 2n) = (T_s f_{2n})(1)$ for $f_{2n}$ the normalized vector in $I_s$ with $K$-type $\rho_{2n}$ also shows that $T_s$ has an analytic continuation on smooth vectors in $I_s$, nor merely $K$-finite vectors, as follows. From $\Gamma(s) \cdot s = \Gamma(s + 1)$ we see that
\[
(T_s f_{2n})(1) = \frac{2\pi 2^{1-2s} \Gamma(2s - 1)}{\Gamma(s-n) \Gamma(s+n)} = \text{polynomial growth in } n
\]
Then let
\[
f = \sum_{n \in \mathbb{Z}} c_{2n} \cdot f_{2n}
\]
be smooth in $I_s$. Smoothness is equivalent to the rapid decrease\[\text{[22]}\] of the Fourier coefficients. Then
\[
T_s f = \sum_{n \in \mathbb{Z}} \lambda(s, 2n) \cdot c_{2n} \cdot f_{2n}
\]
still has rapidly decreasing coefficients, so is a smooth vector in $I_{1-s}$. That is, away from the poles, the intertwining operator $T_s$ when analytically continued is defined on all smooth vectors in $I_{1-s}$, not merely $K$-finite ones.

5. **Appendix: usual tricks with $\Gamma(s)$**

The property of $\Gamma(s)$ used above is standard, but sufficiently important that we review it. The gamma function is given for $\text{Re}(s) > 0$ by Euler’s integral
\[
\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} \frac{dt}{t}
\]
Replacing $t$ by $ty$ with $y > 0$
\[
\Gamma(s) \cdot y^{-s} = \int_0^\infty e^{-ty} t^{s-1} \frac{dt}{t}
\]
By analytic continuation to the right complex half-plane, for $y > 0$ and $x \in \mathbb{R}$
\[
\Gamma(s) \cdot (y + 2\pi i x)^{-s} = \int_0^\infty e^{-t(y + 2\pi i x)} t^{s-1} \frac{dt}{t}
\]
\[\text{[22]}\text{ Rapid decrease means, as usual, that } |n|^N \cdot |c_{2n}| \text{ goes to 0 (as } |n| \rightarrow \infty) \text{ for every } N. \text{ This equivalence is readily proven by repeated integration by parts in the integrals yielding the Fourier coefficients.}\]
Having analytically continued, we may let \( y = 1 \) again, obtaining

\[
\Gamma(s) \cdot (1 + 2\pi ix)^{-s} = \int_0^\infty e^{-t(1+2\pi ix)t^s} \frac{dt}{t} = \int_0^\infty e^{-2\pi ixt} e^{-t^s} \frac{dt}{t}
\]

which is the Fourier transform of

\[
\varphi_s(t) = \begin{cases} 
  e^{-t} t^{s-1} & (t > 0) \\
  0 & (t < 0)
\end{cases}
\]

To compute the concrete integral for \((T_s f)(1)\) we invoke the Plancherel theorem, that

\[
\int_\mathbb{R} f(x) \varphi(x) \, dx = \int_\mathbb{R} \hat{f}(x) \hat{\varphi}(x) \, dx
\]

and Fourier inversion. Then, with real \( s \gg 0 \), replacing \( x \) by \( 2\pi x \) at the first step, and with real \( s \),

\[
\int_\mathbb{R} (1 + ix)^{-s+n} (1 - ix)^{-s-n} \, dx = 2\pi \int_\mathbb{R} (1 + 2\pi ix)^{-s+n} (1 - 2\pi ix)^{-s-n} \, dx
\]

\[
= 2\pi \int_\mathbb{R} \varphi_{s-n}(x) \varphi_{s+n}(x) \, dx = 2\pi \int_\mathbb{R} \varphi_{s-n}(x) \varphi_{s+n}(x) \, dx
\]

\[
= \frac{2\pi}{\Gamma(s - n) \Gamma(s + n)} \int_0^\infty e^{-u} u^{-(s-n)-1} \cdot e^{-u} u^{-(s+n)-1} \, du = \frac{2\pi \Gamma(2s-1)}{\Gamma(s - n) \Gamma(s + n)}
\]
as computed more heuristically earlier. This also exhibits the constant \( 2\pi \).