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Bernstein's continuation principle

Paul Garrett garrett@math.umn.edu http://www.math.umn.edu/˜garrett/

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This is a belatedly corrected version of the continuation-principle parts of [G 2001a] and [G 2001b]. In 2014, J. Hundley kindly observed some sloppiness in the purported proof of the *Banach space criterion* for finite envelope (below). That flawed proof needlessly and falsely asserted too general existence of complementary subspaces in Banach spaces. Repairing that gaffe is the main point of this updated document. There are also some edits without mathematical content.

Regarding the application to meromorphic continuation of Eisenstein series, by now we have [Bernstein-Lapid 2020], which also gives some references to more recent related developments.

[G 2018] systematically develops the relevant functional analysis in chapters 9, 13, 14, and 15, including Gelfand-Pettis vector-valued integrals, and Grothendieck's results about holomorphic vector-valued functions. There is also a substantial historical and bibliographic discussion there. See also [G 2020]

Thanks to L. Carbone for interest in these corrections.

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1. *Weak-to-strong issues*

A function f taking values in a topological vectorspace V is *weakly holomorphic* when $s \to (\lambda \circ f)(s)$ is holomorphic ($\mathbb{C}$-valued) for every $\lambda \in V^*$. A family of operators $T_s : V \to W$ from one topological vectorspace to another is *weakly holomorphic* in a parameter s if for every vector $v \in V$ and for every continuous functional $\mu \in W^*$ the $\mathbb{C}$-valued function $\mu(T_s v)$ is holomorphic in s.

[1.1] Claim: : For $S_s : X \to Y$ and $T_s : Y \to Z$ be two weakly holomorphic families of continuous linear operators on topological vectorspaces X, Y, Z, the composition $T_s \circ S_s : X \to Z$ is weakly holomorphic. For a weakly holomorphic Y-valued function $s \to f(s)$, the composite $T_s \circ f$ is a weakly holomorphic Z-valued function.

Proof: This is a corollary of Hartogs' theorem, that separate analyticity of a function of several complex variables implies joint analyticity (without any other hypotheses). Consider the family of operators $T_t \circ S_s$. By weak holomorphy, for $x \in X$ and a linear functional $\mu \in Z^*$ the $\mathbb{C}$-valued function $(s,t) \to \mu(T_t(S_s(x))$ is separately analytic. By Hartogs' theorem, it is jointly analytic. It follows that the diagonal function $s \to (s,s) \to \mu(T_s(S_s(x))$ is analytic. The second assertion has a nearly identical proof. ///

A *Gelfand-Pettis* or *weak* integral of a function $s \to f(s)$ on a measure space (X, μ) with values in a topological vectorspace V is an element $I \in V$ so that for all $\lambda \in V^*$

$$\lambda(I) \;=\; \int_X (\lambda \circ f)(s)\, d\mu(s)$$

A topological vectorspace is *quasi-complete* when every *bounded* (in the topological vectorspace sense, not necessarily the metric sense) Cauchy *net* is convergent.

[1.2] Theorem: Continuous compactly-supported functions $f : X \to V$ with values in *quasi-complete* (locally convex) topological vectorspaces V have Gelfand-Pettis integrals with respect to finite positive regular Borel measures μ on compact spaces X, and these integrals are *unique*. In particular, for a μ with total measure $\mu(X) = 1$, the integral $\int_X f(x)\, d\mu(s)$ lies in the closure of the convex hull of the image $f(X)$ of the measure space X.

Proof: Bourbaki's *Integration*. (Thanks to Jacquet for bringing this reference to my attention.) ///

The following property of Gelfand-Pettis integrals is broadly useful in applications, such as justifying differentiation under integrals.

[1.3] Claim: Let $T : V \to W$ be a continuous linear map, and let $f : X \to V$ be a continuous compactly supported V-valued function on a topological measure space X with finite positive Borel measure μ. Suppose that V is locally convex and quasi-complete, so that (from above) a Gelfand-Pettis integral of f exists and is unique. Suppose that W is locally convex. Then

$$T\left(\int_X f(x)\, d\mu(x)\right) \;=\; \int_X Tf(x)\, d\mu(x)$$

In particular, $T\left(\int_X f(x)\, d\mu(x)\right)$ is a Gelfand-Pettis integral of $T \circ f$.

Proof: First, the integral exists in V, from above. Second, for any continuous linear functional λ on W, $\lambda \circ T$ is a continuous linear functional on V. Thus, by the defining property of the Gelfand-Pettis integral, for every such λ

$$(\lambda \circ T)\left(\int_X f(x)\, d\mu(x)\right) \;=\; \int_X (\lambda T f)(x)\, d\mu(x)$$

This exactly asserts that $T\left(\int_X f(x)\, d\mu(x)\right)$ is a Gelfand-Pettis integral of the W-valued function $T \circ f$. Since the two vectors $T\left(\int_X f(x)\, d\mu(x)\right)$ and $\int_X Tf(x)\, d\mu(x)$ give identical values under all $\lambda \in W^*$, and since W is locally convex, these two vectors are equal, as claimed. ///

[1.4] Corollary: For quasi-complete and locally convex V, weakly holomorphic V-valued functions are (strongly) holomorphic.

Proof: The Cauchy integral formulas involve continuous integrals on compacta, so these integrals exist as Gelfand-Pettis integrals. Thus, we can obtain V-valued convergent power series expansions for weakly holomorphic V-valued functions, from which (strong) holomorphy follows by an obvious extension of Abel's theorem that analytic functions are holomorphic. ///

Give the space $\mathrm{Hom}^o(X, Y)$ of continuous mappings $T : X \to Y$ from an LF space X (strict colimit of Fréchetspaces, e.g., a Fréchetspace) to a quasi-complete space Y the *weak operator* topology as follows. For $x \in X$ and $\mu \in Y^*$, define a seminorm $p_{x,\mu}$ on $\mathrm{Hom}^o(X, Y)$ by

$$p_{x,\mu}(T) \;=\; |\mu(T(x))|$$

[1.5] Corollary: With the weak topoloogy $\mathrm{Hom}^o(X, Y)$ is quasi-complete.

Proof: The collection of finite linear combinations of the functionals $T \to \mu(Tx)$ is exactly the dual space of $\mathrm{Hom}^o(X, Y)$ with the weak operator topology. Invoke the previous result. ///

[1.6] Corollary: A weakly holomorphic $\mathrm{Hom}^o(X, Y)$-valued function T_s is holomorphic when $\mathrm{Hom}^o(X, Y)$ is given the weak operator topology. ///

[1.7] Remark: $\mathrm{Hom}^o(X, Y)$ is also quasi-complete for certain other topologies, but we do not need that stronger result. See [G 2020].

2. *A continuation principle*

Let V be a topological vector space. Following Bernstein, a *system of linear equations* X_o *in* V is a collection

$$X_o \;=\; \{(W_i, T_i, w_i) : i \in I\}$$

where I is a (not necessarily countable) set of indices, each W_i is a topological vector space,

$$T_i : V \longrightarrow W_i$$

is a continuous linear map for each index i, and $w_i \in W_i$ are the *targets*. A *solution* of the system X_o is $v \in V$ such that $T_i(v) = w_i$ for all indices i. Denote the set of solutions by $\mathrm{Sol}\, X_o$.

When the systems of linear equations $X_s = \{W_i, T_{i,s}, w_{i,s}\}$ depend on a parameter s, with $T_{i,s}$ and $w_{i,s}$ weakly holomorphic in s, say that the *parametrized linear system* $X = \{X_s : s \in S\}$ is *holomorphic* in s. Note that $\{W_i\}$ does not depend upon s.

For $X = \{X_s\}$ a parametrized system of linear equations in a space V, holomorphic in s, suppose there is a finite-dimensional space F, a weakly holomorphic family $\{f_s\}$ of continuous linear maps $f_s : F \to V$ such that, for each s, $\mathrm{Im}\, f_s \supset \mathrm{Sol}\, X_s$ is a *finite holomorphic envelope* for the system X, and X is of *finite type.*

For $U_\alpha, \alpha \in A$ an open cover of the parameter space, and for each $\alpha \in A$ $\{f_s^{(\alpha)} : s \in U_\alpha\}$ is a finite envelope for the system $X^{(\alpha)} = \{X_s : s \in U_\alpha\}$, say that $\{f_s^{(\alpha)} : s \in U_\alpha, \alpha \in A\}$ is a *locally finite holomorphic envelope* of X.

[2.1] Remark: When there is a meromorphic continuation v_s of a solution, by taking $F = \mathbb{C}$ and $f_s : \mathbb{C} \to V$ to be $f_s(z) = z \cdot v_s$ we trivially obtain a finite holomorphic envelope for parameter values s away from the poles of v_s. That is, if there is a meromorphic continuation, then for trivial reasons there is a finite holomorphic envelope, and the system is of finite type.

[2.2] Theorem: (*Bernstein*) *Continuation Principle:* Let $X = \{X_s : s\}$ be a *locally finite* system of linear equations

$$T_{i,s} : V \to W_i$$

for s varying in a connected complex manifold, with each W_i (locally convex and) *quasi-complete.* Then the *continuation principle* holds. That is, if for s in some non-empty open subset there is a unique solution v_s, then this solution depends meromorphically upon s, has a meromorphic continuation to s in the whole manifold, and for fixed s off a locally finite set of analytic hypersurfaces inside the complex manifold, the solution v_s is the *unique* solution to the system X_s.

Proof: This reduces to a holomorphically parametrized version of Cramer's rule, in light of comments above about weak-to-strong principles and composition of weakly holomorphic maps.

It is sufficient to check the continuation principle locally, so let $f_s : F \to V$ be a family of morphisms so that $\mathrm{Im}\, f_s \supset \mathrm{Sol}\, X_s$, with F finite-dimensional. We can reformulate the statement in terms of the finite-dimensional space F. Namely, put

$$K_s^+ = \{v \in F : f_s(v) \in \mathrm{Sol}\, X_s\} = \{\text{ inverse images in } F \text{ of solutions }\}$$

(The set K_s^+ is an affine subspace of F.) By elementary finite-dimensional linear algebra, X_s has a unique solution if and only if

$$\dim K_s^+ \;=\; \dim \ker f_s$$

The weak holomorphy of $T_{i,s}$ and f_s yield the weak holomorphy of the composite $T_{i,s} \circ f_s$ from the finite-dimensional space F to W_i, by the corollary of Hartogs' theorem above. The finite-dimensional space F is certainly LF, and W_i is quasi-complete, so by invocation of results above on weak holomorphy the space $\mathrm{Hom}^o(F, W_i)$ is quasi-complete, and a weakly holomorphic family in $\mathrm{Hom}^o(F, W_i)$ is in fact holomorphic.

Take $F = \mathbb{C}^n$. Using linear functionals on V and W_i which separate points we can describe $\ker f_s$ and K_s^+ by systems of linear equations of the forms

$$\ker f_s = \{(x_1, \ldots, x_n) \in F : \sum_j a_{\alpha\, j}\, x_j = 0, \ \ \alpha \in A\}$$

$$K_s^+ = \{ \text{ inverse images of solutions } \} = \{(x_1, \ldots, x_n) \in F : \sum_j b_{\beta\, j}\, x_j = c_\beta, \ \ \beta \in B\}$$

where $a_{\alpha\, j}$, $b_{\beta\, j}$, c_β all depend implicitly upon s, and are holomorphic $\mathbb{C}$-valued functions of s. (The index sets A, B may be of arbitrary cardinality.) Arrange these coefficients into matrices M_s, N_s, Q_s holomorphically parametrized by s, with entries

$$M_s(\alpha, j) = a_{\alpha\, j} \qquad N_s(\beta, j) = b_{\beta\, j} \qquad Q_s(\beta, j) = \begin{cases} b_{\beta\, j} & \text{for } 1 \le j \le n \\ c_\beta & \text{for } j = n \end{cases}$$

of sizes card(A)-by-n, card(B)-by-n, card(B)-by-$(n+1)$. We have

$$\dim \ker f_s = n - \mathrm{rank}\, M_s$$

Certainly for all s

$$\mathrm{rank}\, N_s \le \mathrm{rank}\, Q_s$$

and if the inequality is *strict* then there is *no solution* to the system X_s. By finite-dimensional linear algebra, assuming that $\mathrm{rank}\, N_s = \mathrm{rank}\, Q_s$,

$$\dim K_s^+ = n - \mathrm{rank}\, B_s$$

Therefore, the condition that $\dim K_s^+ = \dim \ker f_s$ can be rewritten as

$$\mathrm{rank}\, M_s = \mathrm{rank}\, N_s = \mathrm{rank}\, Q_s$$

Let S_o be the dense subset (actually, S_o is the complement of an analytic subset) of the parameter space where $\mathrm{rank}\, M_s$, $\mathrm{rank}\, N_s$, and $\mathrm{rank}\, Q_s$ all take their maximum values. Since by hypothesis $S_o \cap \Omega$ is not empty, and since the ranks are equal for $s \in \Omega$, all those maximal ranks are equal to the same number r. Then for all $s \in S_o$ the rank condition holds and X_s has a solution, and the solution is unique.

In order to prove the continuation principle we must show that $X = \{X_s\}$ has a meromorphic solution v_s. Making use of the finite envelope of the system X, to find a meromorphic solution of X it is enough to find a meromorphic solution of the parametrized system $Y = \{Y_s\}$ where

$$Y_s = \{\sum b_{\beta\, i} x_i = c_\beta : \text{ for all } \beta\}$$

with implicit dependence upon s. Let r be the maximum rank, as above. Choose $s_o \in S_o$ and choose an r-by-r minor

$$D_{s_o} = \{b_{\beta, j} : \beta \in \{\beta_1, \ldots, \beta_r\}, j \in \{j_1, \ldots, j_r\}\}$$

of full rank, inside the matrix N_{s_o}, with constraints on the indices as indicated. Let $S_1 \subset S_o$ be the set of points s where D_s has full rank, that is, where $\det D_s \neq 0$. Consider the system of equations

$$Z = \{ \sum_{j \in \{j_1, \ldots, j_r\}} b_{\beta\, j}\, x_j = c_\beta : \beta \in \{\beta_1, \ldots, \beta_r\}\} \quad (\text{with } s \text{ implicit})$$

By Cramer's Rule, for $s \in S_1$ this system has a unique solution $(x_{1,s}, \ldots, x_{r,s})$. Further, since the coefficients are holomorphic in s, the expression for the solution obtained via Cramer's rule show that the solution is meromorphic in s. Extending this solution by $x_j = 0$ for j not among $j_1, \ldots, j_r$, we see that it satisfies the r equations corresponding to rows $\beta \in \{\beta_1, \ldots, \beta_r\}$ of the system Y_s. Then for $s \in S_1$ the equality $\mathrm{rank}\, N_s = \mathrm{rank}\, Q_s = r$ implies that after satisfying the first r equations of Y_s it will automatically satisfy the rest of the equations in the system Y_s.

Thus, the system has a *weakly* holomorphic solution. Earlier observations on weak-to-strong principles assure that this solution is holomorphic. This proves the continuation principle. ///

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3. *Finite envelope criteria*

[3.1] Claim: (*Dominance*) (Called *inference* by Bernstein.) Let $X' = \{X'_s\}$ be another holomorphically parametrized system of equations in a linear space V', with the same parameter space as a system $X = \{X_s\}$ on a space V. We say that X' *dominates* X when there is a family of morphisms $h_s : V' \to V$, weakly holomorphic in s, so that

$$\mathrm{Sol}\, X_s \ \subset \ h_s(\mathrm{Sol}\, X'_s) \qquad\qquad (\text{for all } s)$$

If X'_s is locally finite then X_s is locally finite.

Proof: The fact that compositions of weakly holomorphic mappings are weakly holomorphic assures that X'_s really meets the definition of *system*. Granting this, the conclusion is clear. ///

[3.2] Theorem: (*Banach-space criterion*) Let V be a Banach space, and X a single parametrized homogeneous equation $T_s(v) = 0$, with $T_s : V \to W$, where W is also a Banach space, and where $s \to T_s$ is holomorphic for the uniform-norm Banach-space topology on $\mathrm{Hom}^o(V, W)$. If for some fixed s_o there exists an operator $A : W \to V$ so that $A \circ T_{s_o}$ has *finite-dimensional kernel* and *closed image*, then X_s is of *finite type* in some neighborhood of s.

Proof: Let V_1 be the image of $A \circ T_{s_o}$, and V_o the kernel of $A \circ T_{s_o}$.

We claim that finite dimensional $V_o \subset V$ has a continuous linear $p : V \to V_o$ which is the identity on V_o. Indeed, for a basis $v_1, \ldots, v_n$ of V_o, and for $v \in V_o$, the coefficients $c_i(v)$ in the expression $v = \sum_i c_i(v) v_i$ are continuous linear functionals on V_o. By Hahn-Banach, each c_i extends to a continuous linear functional λ_i on V, and $p(v) = \lambda_1(v)v_1 + \ldots + \lambda_n(v)v_n$ is as desired.

Let $q = A \circ T_{s_o} : V \to V_1$.

Let X'_s be a new system in V, given by a single equation $T'_s(v) = 0$, where $T'_s = q \circ T_s : V \to V_1$. If $T_s(v) = 0$, then $T'_s(v) = 0$, so X'_s *dominates* X_s.

Since $V_1 \subset V$ is *closed*, it is a Banach space. Consider the holomorphic family of maps

$$\varphi_s \ = \ p \oplus T'_s \ : \ V \longrightarrow V_o \oplus V_1$$

where $V_o \oplus V_1$ is given its natural Banach space structure. The function $s \to \varphi_s$ is holomorphic for the operator-norm topology on $\mathrm{Hom}^o(V, V_o \oplus V_1)$.

By construction, φ_{s_o} is a bijection, so by the Open Mapping Theorem it is an isomorphism. The continuous inverse $\varphi_{s_o}^{-1}$ has an operator norm δ^{-1} with $0 < \delta^{-1} < +\infty$. With s sufficiently near s_o so that $|\varphi_{s_o} - \varphi_s| < \delta/2$,

$$|\varphi_s(x)| \ \geq \ |\varphi_{s_o}(x)| - |\varphi_s(x) - \varphi_{s_o}(x)| \ \geq \ \delta \cdot |x| - \frac{\delta}{2} \cdot |x| \ \geq \ \frac{\delta}{2} \cdot |x|$$

Thus, φ_s is an isomorphism for s sufficiently near s_o.

The map $s \to \varphi_s^{-1}$ is holomorphic on a neighborhood of s_o, since the operator-norm topology restricted to invertible elements in $\mathrm{Hom}^o(V, V_o \oplus V_1)$ is the same as the operator-norm topology restricted to invertible elements in $\mathrm{Hom}^o(V_o \oplus V_1, V)$. This follows from the continuity of $T \to T^{-1}$ on a neighborhood of an invertible operator.

There is a finite envelope $\varphi_s^{-1}(V_o \oplus \{0\})$ for X'_s. By *dominance*, there is a finite envelope for X_s. ///

[3.3] Corollary: *(Compact operator criterion)* Let V be a Banach space with system X_s given by a single equation $T_s : V \to W$, with Banach space W, requiring $T_s(v) = 0$, with $s \to T_s$ holomorphic for the operator-norm topology. Suppose for some s_o the operator T_{s_o} has a left inverse modulo compact operators, that is, there exists $A : W \to V$ such that

$$A \circ T_{s_o} \;=\; 1_V + (\text{compact operator})$$

Then X_s is of finite type in some neighborhood of s_o.

Proof: Let K be that compact operator. The kernel $V_o = \ker(1_V + K)$ is the -1 eigenspace for K, finite-dimensional by the spectral theory of compact (not necessarily self-adjoint or normal) operators. Similarly, the image V_1 is closed. Thus, the theorem applies. ///

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