Designed pseudo-Laplacians

E. Bombieri P. Garrett

February 15, 2020

Abstract

We elaborate and make rigorous various speculations about the implications of spectral properties of self-adjoint operators on spaces of automorphic forms for location of zeros of L-functions. Some of these ideas arose in work of Colin de Verdière, Lax-Phillips, and Hejhal, from the late 1970s and early 1980s, not to mention semi-apocryphal attributions to Polya and Hilbert. For example, given a complex quadratic extension k of $\mathbb{Q}$, we give a natural self-adjoint extension of a restriction of the invariant Laplacian on the modular curve whose discrete spectrum, if any, consists of values $s(s-1)$ for zeros s of $\zeta_k(s)$. Unfortunately, there seems to be no reason for this discrete spectrum to be large. In fact, Montgomery's pair correlation, and the behavior of $\zeta(1+it)$, imply that at most 94% of zeros of $\zeta(s)$ can appear in this discrete spectrum. Less naively, some preliminary positive results about the dynamics of zeros do follow from these considerations.

1 Introduction

First, a simple example of the content of theorem 10: there is a Hilbert space $\mathfrak{E}^0$ (described just below) of automorphic forms on $SL_2(\mathbb{Z})\backslash\mathfrak{H}$ such that, for every complex quadratic field extension k of $\mathbb{Q}$, there is a self-adjoint extension $\widetilde{S}_k$ of a certain restriction S_k of $-\Delta$, with the invariant Laplace-Beltrami operator $\Delta =$

$y^2(\partial^2/\partial x^2+\partial^2/\partial y^2)$, such that the eigenvalues $\lambda_s = s(1-s) > 1/4$ of $\widetilde{S}_k$ on $\mathfrak{E}^0$, *if any*, occur only for s a zero of the Dedekind zeta function $\zeta_k(s)$. Something like this is suggested by parts of [5], although perhaps often misunderstood as heuristics rather than as potentially rigorous arguments. In any case, this is not what one might hope: there is no assertion of existence of *any* eigenvalues of $\widetilde{S}_k$, and there is no assertion that every zero of $\zeta_k(s)$ gives an eigenvalue. Corollary 69 shows that, assuming Montgomery's pair correlation conjecture [21], at most 94% of the zeros s of $\zeta(s)$ can occur as parameters for eigenvalues λ_s of any particular $\widetilde{S}_k$, suggesting that perhaps *none* do.

Eigenvalues of self-adjoint operators are *real*. Thus, when parametrized as $\lambda_s = s(1-s) \in \mathbb{R}$, either $\Re(s) = \frac{1}{2}$ or $s \in \mathbb{R}$. Thus, as apocryphally suggested by G. Polya and by D. Hilbert, one might imagine proving the Riemann Hypothesis by finding a self-adjoint operator such that, for every non-trivial zero s of $\zeta(s)$, λ_s is an eigenvalue. (See [24].)

In [20] an argument is sketched for discrete decomposition (i.e., pure point spectrum) of $L^2_a(\Gamma\backslash\mathfrak{H})$, the latter space defined to be L^2 automorphic forms with constant term vanishing at height $y \geq a > 1$, with respect to the Friedrichs self-adjoint extension $\widetilde{S}_\Theta$ of the restriction S_Θ of $-\Delta$ to $C^\infty_c(\Gamma\backslash\mathfrak{H}) \cap L^2_a(\Gamma\backslash\mathfrak{H})$. The first surprise is that part of the *continuous* spectrum of the original invariant Laplace-Beltrami operator lying inside $L^2_a(\Gamma\backslash\mathfrak{H})$, consisting of suitable wave-packets of Eisenstein series E_s, becomes discrete. Even more surprising, the new eigenfunctions with eigenvalues $\lambda > 1/4$ are certain *truncated Eisenstein series* $\wedge^a E_s$, namely, for s such that $a^s + c_s a^{1-s} = 0$, where $a^s + c_s a^{1-s}$ is the constant term of E_s evaluated at height a, seemingly contradicting elliptic regularity. Evidently eigenfunctions of the Friedrichs extension of a restriction of an elliptic operator can fail to be *smooth*. Thus, $a^s + c_s a^{1-s} = 0$ if and only if $\lambda_s = s(1-s)$ is an eigenvalue of $\widetilde{S}_\Theta$. By self-adjointness, $a^s + c_s a^{1-s} = 0$ only for $s \in \frac{1}{2} + i\mathbb{R}$ (or in $[0,1]$). This idea appears in [20], pages 204-6.

In that context, the document [14] was provocative: attempting numerical determination of eigenvalues for $-\Delta$ on $SL_2(\mathbb{Z})\backslash\mathfrak{H}$, the list of spectral parameters s for eigenvalues $\lambda_s = s(1-s)$ was observed by H. Stark and D. Hejhal to include low-lying zeros of $\zeta(s)$ and of $L(s,\chi_{-3})$ with the Dirichlet L-function of conductor 3. [16] reported attempted reproduction of the numerical results, with the zeros of zeta and the L-function notably missing from the list of spectral parameters, and observing that the spurious appearance of these values in the list of [14] was due to a mis-application of the Henrici collocation method [8]. Hejhal further observed that the solution procedure of [14] in fact allowed as solution a value $u(z) = G_{\lambda_s}(z,\omega)$ of an *automorphic Green's function* $G_{\lambda_s}(z,z_o)$ as a solution, with $\omega = e^{2\pi i/6}$. Automorphic Green's functions and related meromorphic families had been studied in [22], [23], and [7].

That is, the values $\lambda_s = s(1-s)$ obtained by [14] *not* belonging to genuine cuspforms were values λ_s fitting into an equation $(-\Delta - \lambda_s)u = \delta_\omega^{\mathrm{afc}}$ for Γ-invariant Dirac $\delta_\omega^{\mathrm{afc}}$ on $\mathfrak{H}$, supported on images of $\omega = e^{2\pi i/6}$. Since λ_s appearing in such an equation are not necessarily eigenvalues of a self-adjoint operator, they need not be *real*. A claim that $\delta_\omega^{\mathrm{afc}}$ can be disregarded on the grounds that it has support of measure zero is incorrect. Nevertheless, there were precedents in [20], [4], and [5] for legitimate reinterpretation of certain inhomogeneous equations as homogeneous equations. For example, in [20] the exotic eigenfunctions for $\lambda > 1/4$ (certain truncated Eisenstein series) are solutions of an *inhomogeneous* distributional equation $(-\Delta - \lambda_w)u = \eta_a$, where η_a evaluates the constant term of an automorphic form at height $y = a$. This was a precedent for the suggestion at the end of [5] that the inhomogeneous equation $(-\Delta - \lambda_s)u = \delta_\omega^{\mathrm{afc}}$ on $\Gamma\backslash\mathfrak{H}$ could be converted to a homogeneous equation involving a self-adjoint operator, so that the values λ_s would become (genuine) eigenvalues, and thus be *real*.

A complication observed in [5] for an equation $(-\Delta - \lambda_s)u = \theta$ on $\Gamma\backslash\mathfrak{H}$, with distribution θ on $\Gamma\backslash\mathfrak{H}$, is that the Friedrichs extension $\widetilde{S}_\theta$ of the restriction S of $-\Delta$ to the kernel of θ on $C_c^\infty(\Gamma\backslash\mathfrak{H})$ converts this inhomogeneous equation to a homogeneous one with an auxiliary condition, only for θ in a suitable (global automorphic) Sobolev space $H^{-1}(\Gamma\backslash\mathfrak{H})$. This is the Hilbert-space dual of $H^1(\Gamma\backslash\mathfrak{H})$, which is the completion of $C_c^\infty(\Gamma\backslash\mathfrak{H})$ with respect to the Sobolev H^1 norm given by $|f|^2_{H^1} = \big\langle (1-\Delta)f, f\big\rangle_{L^2}$ for $f \in C_c^\infty(\Gamma\backslash\mathfrak{H})$, discussed in subsection 3.4.

To solve equations $(-\Delta - \lambda)u = \theta$ with distributions θ, it is advantageous to use a *spectral characterization* of (global automorphic) Sobolev spaces. Recall the spectral decomposition of $L^2(\Gamma\backslash\mathfrak{H})$ with $\Gamma = SL_2(\mathbb{Z})$, for example from [6] or [18]: first, for $f \in C_c^\infty(\Gamma\backslash\mathfrak{H})$,

$$f \;=\; \sum_F \langle f, F\rangle \cdot F + \frac{\langle f, 1\rangle \cdot 1}{\langle 1, 1\rangle} + \frac{1}{4\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \langle f, E_s\rangle \cdot E_s\, ds$$

where F runs over an orthonormal basis of L^2 cuspforms, with the L^2 hermitian product $\langle,\rangle$ notation abused to denote the corresponding integral for $f \in C_c^\infty(\Gamma\backslash\mathfrak{H})$:

$$\langle f, E_s\rangle \;=\; \int_{\Gamma\backslash\mathfrak{H}} f(z)\cdot \overline{E_s(z)}\,\frac{dx\,dy}{y^2}$$

Then one proves a Plancherel theorem for test functions f:

$$|f|^2 \;=\; \sum_F |\langle f, F\rangle|^2 + \frac{|\langle f, 1\rangle|^2}{\langle 1, 1\rangle} + \frac{1}{4\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} |\langle f, E_s\rangle|^2 \, ds$$

Further, $f \to (s \to \langle f, E_s\rangle)$ has image dense in the space of L^2 functions g on $\frac{1}{2}+i\mathbb{R}$ such that $g(1-s) = c_s \cdot g(s)$. As with the Plancherel-Fourier transform on $L^2(\mathbb{R})$, the spectral expansion extends to a Plancherel theorem for $L^2(\Gamma\backslash\mathfrak{H})$, with the pairings $\langle f, E_s\rangle$ necessarily interpreted as isometric extensions. The spectral synthesis integrals $\int_{(\frac{1}{2})}\langle f, E_s\rangle \cdot E_s \, ds$ converge only in an L^2 sense, certainly not necessarily pointwise. Then, for $r \in \mathbb{R}$ we can define (global automorphic) Sobolev spaces $H^r(\Gamma\backslash\mathfrak{H})$ to be the completion of $C_c^\infty(\Gamma\backslash\mathfrak{H})$ with respect to the H^r norms

$$|f|^2_{H^r} \;=\; \sum_F |\langle f, F\rangle|^2 \cdot (1+\lambda_{s_F})^r + \frac{|\langle f, 1\rangle|^2 \cdot (1+\lambda_1)^r}{\langle 1, 1\rangle}$$

$$+\frac{1}{4\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} |\langle f, E_s\rangle|^2 \cdot (1+\lambda_s)^r \, ds$$

where $s_F \in \mathbb{C}$ gives the eigenvalue λ_{s_F} of cuspform F with respect to Δ, first defined for $f \in C_c^\infty(\Gamma\backslash\mathfrak{H})$. See subsection 3.4.

An obstacle appears: subsection 3.5 observes that an automorphic Dirac δ is not in $H^{-1}(\Gamma\backslash\mathfrak{H})$, but only in $H^{-1-\varepsilon}(\Gamma\backslash\mathfrak{H})$ for every $\varepsilon > 0$. A possible way around this obstacle appears briefly at the end of [5], to consider the restriction θ of $\delta_\omega^{\mathrm{afc}}$ to a smaller Hilbert space of automorphic forms. In [5], this smaller space is suggested to be the orthogonal complement to the discrete spectrum, but it turns out to be necessary to retain the constants, which appear as square-integrable residues of Eisenstein series at $s = 1$. For precision, a different description of the restriction θ is appropriate, as given in detail in section 3.4. First, the suitable analogue of test functions here is the space $\mathfrak{E}_c^\infty$ of pseudo-Eisenstein series

$$\Psi_\varphi(z) \;=\; \sum_{\Gamma_\infty\backslash\Gamma} \varphi(\Im(\gamma z))$$

with test-function data $\varphi \in C_c^\infty(0,\infty)$. The r-th Eisenstein-Sobolev space $\mathfrak{E}^r$ is the completion of $\mathfrak{E}_c^\infty$ with respect to the H^r-norm. Let S be the restriction of $-\Delta$ to $\mathfrak{E}_c^\infty$. Initially, let θ be the restriction of $\delta_{z_o}^{\mathrm{afc}}$ to $\mathfrak{E}_c^\infty$. As we see just below, this restriction θ has finite H^{-1}-norm, so extends continuously to an element of

$\mathfrak{E}^{-1}$. Then take S_θ to be the further restriction of S to have domain $\mathfrak{E}_c^\infty \cap \ker\theta$, which is still dense in $\mathfrak{E}^0$, since $\theta \notin \mathfrak{E}^0$. Friedrichs' construction applies to the unbounded operator S_θ on the Hilbert space $\mathfrak{E}^0$. In present terms, for the analogous restriction θ of a finite real-linear combination of automorphic δ such that $\theta E_s = (\sqrt{d_k}/2)^s \cdot \zeta_k(s)/\zeta(2s)$ for d_k the (absolute value of) the discriminant of k, since θ is a restriction of a distribution of compact support on $\Gamma\backslash\mathfrak{H}$, θ is in $\mathfrak{E}^{-\infty} = \bigcup_r \mathfrak{E}^r$. Its H^r-norm would be

$$|\theta|^2_{H^r} \;=\; \frac{|\langle\theta,1\rangle|^2\cdot(1+\lambda_1)^r}{\langle 1,1\rangle} + \frac{1}{4\pi i}\int_{(\frac{1}{2})} |(\sqrt{d_k}/2)^s\zeta_k(s)/\zeta(2s)|^2\cdot(1+\lambda_s)^r\;ds$$

By the second-moment bound of [15] and Landau's bounds on the behavior of $\zeta(s)$ on the edge of the critical strip, the integral is finite for $r < -3/4$. Thus, $\theta \in \mathfrak{E}^{-\frac{3}{4}-\varepsilon} \subset \mathfrak{E}^{-1}$, and has a spectral expansion converging in $\mathfrak{E}^{-1}$:

$$\theta \;=\; \frac{\langle\theta,1\rangle\cdot 1}{\langle 1,1\rangle} + \frac{1}{4\pi i}\int_{(\frac{1}{2})} \frac{a_k^{1-s}\zeta_k(1-s)}{\zeta(2s)}\cdot E_s\;ds$$

Certainly this does not converge pointwise, but does converge in $\mathfrak{E}^{-1}$. Such expansions allow rigorous solution of differential equations $(-\Delta-\lambda_w)u = \theta$ *by division*: for $\Re(w) > \frac{1}{2}$ and $w \notin \mathbb{R}$, a solution in $\mathfrak{E}^1$ is

$$u_{\theta,w} \;=\; \frac{\langle\theta,1\rangle\cdot 1}{(\lambda_1-\lambda_w)\cdot\langle 1,1\rangle} + \frac{1}{4\pi i}\int_{(\frac{1}{2})} \frac{a_k^{1-s}\zeta_k(s)}{\zeta(2s)\cdot(\lambda_s-\lambda_w)}\cdot E_s\;ds$$

Since $\theta \in \mathfrak{E}^{-1}$, the Friedrichs extension $\widetilde{S}_\theta$ of the restriction S_θ of $-\Delta$ to $\mathfrak{E}_c^\infty \cap \ker\theta$ behaves as desired: for $u \in V$, $(\widetilde{S}_\theta - \lambda)u = 0$ if and only if $(-\Delta-\lambda)u = c\cdot\theta$ for some constant c, and $\theta u = 0$. That is, the inhomogeneous distributional equation $(-\Delta-\lambda)u = c\cdot\theta$ is equivalent to a homogeneous equation $(\widetilde{S}_\theta-\lambda)u = 0$ together with the auxiliary condition $\theta u = 0$.

At this point, we can make precise sense of one speculation from [5]: with $\theta \in \mathfrak{E}^{-1}$ the restriction of $\delta_\omega^{\mathrm{afc}}$ to $\mathfrak{E}^1$ (after extending by continuity), theorem 10 shows that the discrete spectrum $\lambda_s > 1/4$ of $\widetilde{S}_\theta$, *if any*, is $\lambda_s = s(1-s)$ with s a zero of $\zeta(s)\cdot L(s,\chi_{-3})$. There is no assurance of existence of any discrete spectrum.

Spectral theory can be applied in less naive ways: systematic construction of natural self-adjoint operators $S \geq 0$ exhibits meromorphic functions whose zeros

s are on the critical line $\frac{1}{2} + i\mathbb{R}$, and perhaps also on $[0,1]$. The arguably next-simplest continuation is to consider *two* elements $\eta, \theta \in \mathfrak{E}^{-1}$, such that

$$(\mathbb{C} \cdot \eta + \mathbb{C} \cdot \theta) \cap \mathfrak{E}^0 \;\; = \;\; \{0\}$$

with $\eta^c = \eta$ and $\theta^c = \theta$, the restriction $S_{\eta,\theta}$ of $-\Delta$ to domain $\widetilde{S}_{\eta,\theta}$. Let $u_{\eta,w}$ and $u_{\theta,w}$ be solutions in $\mathfrak{E}^1(\Gamma\backslash\mathfrak{H})$ of $(-\Delta - \lambda_w)u = \eta$ and $(-\Delta - \lambda_w)u = \theta$, respectively. The off-line non-vanishing argument (see subsection 4.4) shows that for $\Re(w) > \frac{1}{2}$ and $w \notin \mathbb{R}$,

$$\langle \eta, u_{\eta,w} \rangle \cdot \langle \theta, u_{\theta,w} \rangle - \langle \eta, u_{\theta,w} \rangle \cdot \langle \theta, u_{\eta,w} \rangle \;\; \neq \;\; 0$$

where the pairings are on $\mathfrak{E}^{-1} \times \mathfrak{E}^1$. Taking η to be the constant-term evaluation

$$\eta_a f \;\; = \;\; \int_0^1 f(x + ia)\, dx$$

with $a \gg_\theta 1$ gives (see subsection 4.4 and the prior calculations in section 5)

$$\langle \eta_a, u_{\theta,w} \rangle \;\; = \;\; \frac{\theta E_w}{2w - 1} \;\; = \;\; \langle \theta, u_{\eta_a,w} \rangle$$

For complex quadratic k over $\mathbb{Q}$ with absolute value of discriminant d_k, let $a_k = \sqrt{d_k}/2$, take $\theta \in \mathfrak{E}^{-1}$ such that $\theta E_w = a_k^w \zeta_k(w)/\zeta(2w)$. Then the non-vanishing assertion becomes explicit: in $\Re(w) > \frac{1}{2}$ and $w \notin (\frac{1}{2}, 1]$, for $a \geq a_k$, after some natural simplifications,

$$a_k^{1-2w} \cdot (a^{2w-1} + c_w) \cdot \Big(\frac{h_k^2/a_k}{\lambda_1 - \lambda_w} + \frac{1}{4\pi i} \int_{(\frac{1}{2})} \Big| \frac{\zeta_k(s)}{\zeta(2s)} \Big|^2 \frac{ds}{\lambda_s - \lambda_w} \Big)$$
$$- \frac{1}{2w - 1} \cdot \frac{\zeta_k(w)^2}{\zeta(2w)^2} \;\; \neq \;\; 0$$

where h_k is the class number of k. In fact, meromorphic continuation and functional equation Theorem 29 and Corollary 30 show that the latter expression has zeros only on $\Re(w) = \frac{1}{2}$ and $[0,1]$.

Returning to the simplest situation $\theta \in \mathfrak{E}^{-1}$, with $\theta E_w = a_k^w \zeta_k(w)/\zeta(2w)$, definitive proof of presence or absence of discrete spectrum for the operator $\widetilde{S}_\theta$ seems difficult. Apart from numerical tests, which suggest that there is *no* discrete spectrum, some clarification can be achieved by a subtler application of operator theory and distribution theory, as follows. By direct computation, the constant term of a solution u_w to the distributional equation $(-\Delta - \lambda_w)u = \theta$ vanishes at height $y \gg_\theta 1$. Thus, such u_w lies inside the corresponding Lax-Phillips space $L^2_a(\Gamma\backslash\mathfrak{H})$ (as above, and as in section 4.2), and can be expanded in terms of the (exotic) eigenfunctions for $\widetilde{S}_\Theta$ (mostly certain truncated Eisenstein series). Expanding (the image of) θ in terms of those eigenfunctions, we find (see subsection 7.3) an *interleaving property*: there is at most one spectral parameter w for an eigenvalue of $\widetilde{S}_\theta$ between any two adjacent zeros s of $a^s + c_s a^{1-s}$. Arguing as in [1], the *average* vertical spacing of zeros of $\zeta_k(s)$ (on the critical line or not) at height T is $\pi/\log T$, the same as that of the spacing of zeros of $a^s + c_s a^{1-s}$, which bodes well. However, from [27] (5.17.4) page 112 (in an earlier edition, page 98), for $\log\log T$ large, the argument of $\zeta(s)$ on the edge of the critical strip is relatively regular, so eventually the spacing of the zeros of $a^s + c_s a^{1-s}$ is similarly regular. That is, given $\varepsilon > 0$, for $\log\log T$ sufficiently large, the space between consecutive zeros of $a^s + c_s a^{1-s}$ is between $(1-\varepsilon)\pi/\log T$ and $(1+\varepsilon)\pi/\log T$. Adjusting a slightly, the interleaving property shows that, given $\varepsilon > 0$, for $\log\log T$ sufficiently large the space between consecutive zeros of $\zeta_{\mathbb{Q}(\omega)}(s)$ on the critical line is at least $(1-\varepsilon)\pi/\log T$. This would be in conflict with the pair correlation conjecture [21]: for example, under the Riemann Hypothesis and assuming pair correlation, such a lower bound would allow at most 94% of zeros s of $\zeta(s)$ to appear as parameters for eigenvalues λ_s (see corollary 69).

Corollary 76 proves an illustrative positive result about spacing of on-the-line zeros of $\zeta_k(w)$, *without* any assumptions about point spectrum of self-adjoint operators. Namely, let $t < t'$ be large, and such that $\frac{1}{2} + it$ and $\frac{1}{2} + it'$ are adjacent on-line zeros of $\zeta_k(w)$. Take $\theta \in \mathfrak{E}^{-1}$ such that $\theta E_w = a_k^w \zeta_k(w)/\zeta(2w)$. Suppose that neither $\frac{1}{2} + it$ nor $\frac{1}{2} + it'$ is a zero of

$$J_{\theta,w} \;=\; \frac{h_k^2}{(\lambda_1 - \lambda_w)\cdot\langle 1, 1\rangle} + \frac{1}{4\pi i}\int_{(\frac{1}{2})} \frac{|\theta E_s|^2 - |\theta E_w|^2}{\lambda_s - \lambda_w}\,\mathrm{d}s$$

Suppose there is a *unique* zero $\frac{1}{2} + i\tau_o$ of $J_{\theta,\frac{1}{2}+i\tau}$ between $\frac{1}{2} + it$ and $\frac{1}{2} + it'$, and $\frac{\partial}{\partial\tau} J_{\theta,\frac{1}{2}+i\tau} > 0$. Then $|t' - t| \ge \frac{\pi}{\log t}\cdot(1 + O(\frac{1}{\log\log t}))$. That is, in this configuration, the distance between consecutive zeros of $\zeta_k(w)$ must be *at least* the average.

Analogous discussions with similar proofs apply to a broad class of self-adjoint operators on spaces of automorphic forms.

2 Friedrichs extensions

This section deals with the construction and properties of self-adjoint Friedrichs extensions of operators on a complex Hilbert space.

2.1 Friedrichs self-adjoint extensions

Consider complex Hilbert spaces V with inner product $\langle\ ,\ \rangle$ and required to have a complex-conjugate linear *conjugation* map $v \to \overline{v}$, with expected properties:

$$\overline{\overline{v}} = v, \quad \overline{\alpha \cdot v} = \overline{\alpha} \cdot \overline{v} \qquad (\alpha \in \mathbb{C}), \quad \langle v, \overline{w} \rangle = \langle w, \overline{v} \rangle.$$

Spaces of L^2-complex-valued functions on measure spaces, for example, have natural conjugations given simply by pointwise conjugation of functions.

Let S be an unbounded, symmetric, densely-defined operator on V with domain D dense in V. Assume S is *semi-bounded*, specifically, that

$$\inf_{x \in D,\, \langle x,x \rangle = 1} \langle Sx, x \rangle \geqslant c > 0.$$

Suppose D is stabilized by conjugation and that S commutes with the conjugation $v \to \overline{v}$. For $x,\, y \in D$ let us write $\langle x, y \rangle_1 = \langle Sx, y \rangle$ and let $i : D \longrightarrow V_1$ denote the completion of D with respect to the new inner product $\langle x, y \rangle_1$. The space V_1 has a canonical continuous linear map $j : V_1 \to V$ extending by continuity the identity map $D \to D$, because $\langle x, x \rangle_1 \geqslant c\langle x, x \rangle$ for $x \in D$. In fact, j is injective: $\langle w, iv \rangle_1 = \langle jw, Tv \rangle$ for $w \in V_1$ and $v \in D$, so $jw = 0$ implies that w is orthogonal to the image of D in V_1, which is dense. Whenever possible we suppress the inclusions i and j from the notation. We write $\mathrm{inc} = j \circ i$, in a commutative diagram

$$\begin{tikzcd}
V_1 \arrow[r, hook, "j"] & V \\
D \arrow[u, hook, "i"] \arrow[ur, hook, "\mathrm{inc}"'] &
\end{tikzcd}$$

Now assume S is genuinely unbounded, so that $V_1 \neq V$. Recall that the *Friedrichs extension* $(S, D)^\sim$ of the pair (S, D) is a new self-adjoint operator $\widetilde{S} : V_1 \longrightarrow V$ with a new domain $\widetilde{D} \longrightarrow V_1$, extending S in the sense that there is a diagram for the composition $\mathrm{inc} = j \circ i$, in which (only) the outer curvilinear triangle commutes:

$$\begin{tikzcd}
\widetilde{D} \arrow[r, hook, "\mathrm{inc}"'] \arrow[rr, bend left, "(S,D)^\sim"] & V_1 \arrow[r, hook, "\mathrm{j}"'] & V \\
D \arrow[u, hook] \arrow[ur, hook, "i"] \arrow[urr, "(S,D)"'] & &
\end{tikzcd}$$

For simplicity, usually we shall write only $\widetilde{S}$ without mention of the domain $\widetilde{D}$, but the domain $\widetilde{D}$ is part of the description of the Friedrichs extension.

The Friedrichs extension is characterized by its inverse $\widetilde{S}^{-1}$ being an everywhere defined, continuous, self-adjoint operator $\widetilde{S}^{-1} : V \to V_1$, with the $\langle\ ,\ \rangle_1$ topology on V_1, with the property

$$\langle x, \widetilde{S}^{-1} y\rangle_1 = \langle jx, y\rangle \qquad (x \in V_1,\, y \in V)$$

with j the embedding $j : V_1 \to V$ defined before. Thus $\widetilde{S}$ is self-adjoint, $\widetilde{S} \geqslant S$, and

$$\inf_{x\in D,\, \langle x,x\rangle=1} \langle Sx, x\rangle \leqslant \inf_{x\in \widetilde{D},\, \langle x,x\rangle=1} \langle \widetilde{S}x, x\rangle.$$

When $\widetilde{S} \neq S$ the spectra of $\widetilde{S}$ and S can be different. If the spectrum of S is discrete and (λ_ν) is the associate sequence of eigenvalues, and similarly for $\widetilde{S}$, we have $\lambda_\nu \leqslant \widetilde{\lambda}_\nu$ for all ν.

2.2 Friedrichs extensions of restrictions

Let V_{-1} be the complex-linear dual of V_1, with norm

$$|\mu|_{-1} := \sup_{x\in V_1,\, \langle x,x\rangle\leqslant 1} |\mu(x)|.$$

Since V_1 is a Hilbert space, the norm $|\cdot|_{-1}$ gives an inner product $\langle\ ,\ \rangle_{-1}$ by polarization, and V_{-1} is a Hilbert space. Using the conjugation map on V, let $\Lambda : V \to V^*$ be the complex-linear isomorphism of V with its complex-linear dual by means of $\Lambda(x)(y) = \langle y, \overline{x}\rangle = \langle x, \overline{y}\rangle$.

The inclusion $j : V_1 \to V$ dualizes to $j^* : V^* \to V_1^* = V_{-1}$ by means of $(j^*\mu)(x) = \mu(jx)$ for $\mu \in V^*$ and $x \in V_1$. Thus we have

$$\begin{tikzcd} V_1 \arrow[r, hook, "j"] \arrow[rrr, bend right] & V \arrow[r, "\Lambda", "\approx"'] & V^* \arrow[r, "j^*"] & V_{-1} \end{tikzcd}$$

Conjugation acts on V_{-1} by $\overline{\lambda}(x) = \lambda(\overline{x})$.

Define a continuous linear $S^{\#} : V_1 \longrightarrow V_{-1}$, with $\langle \, , \, \rangle_1$ and $\langle \, , \, \rangle_{-1}$ topologies, respectively, by

$$S^{\#}(x)(y) = \langle x, \overline{y} \rangle_1 \qquad (x,\, y \in V_1).$$

By the Riesz-Fréchet theorem, $S^{\#}$ is a topological isomorphism.

Proposition 1. *The restriction of $S^{\#}$ to the domain of $\widetilde{S}$ is $j^* \circ \Lambda \circ \widetilde{S}$. The domain of $\widetilde{S}$ is $\widetilde{D} = \{x \in V_1 \, : \, S^{\#}x \in (j^* \circ \Lambda)V\}$.*

Proof. By construction of the Friedrichs extension, its domain is $\widetilde{D} = \widetilde{S}^{-1}V$. Thus, for $x = \widetilde{S}^{-1}x'$ with $x' \in V$ and all $y \in V_1$ we have

$$\begin{aligned}(S^{\#}x)(y) &= (S^{\#}\widetilde{S}^{-1}x')(y) = \langle \widetilde{S}^{-1}x', \overline{y} \rangle_{-1} = \langle x, \overline{y} \rangle \\ &= ((j^* \circ \Lambda)x')(y) = ((j^* \circ \Lambda \circ \widetilde{S})x)(y).\end{aligned}$$

This shows that $S^{\#}\widetilde{D} = (j^* \circ \Lambda \circ \widetilde{S})\widetilde{D}$.

On the other hand, for $S^{\#}x = (j^* \circ \Lambda)y$ with $y \in V$ we have, for all $z \in V_1$:

$$\langle z, \overline{x} \rangle_1 = (S^{\#}x)(z) = ((j^* \circ \Lambda)y)(z) = (\Lambda y)(jz) = (jz, \overline{y}) = \langle z, \widetilde{S}^{-1}\overline{y} \rangle_1.$$

Therefore, $\overline{x} = \widetilde{S}^{-1}\overline{y}$, proving the second statement of the proposition. *Q.E.D.*

Let Θ be a finite-dimensional subspace of V_{-1} with

$$\Theta \cap (j^* \circ \Lambda)V = \{0\}. \tag{2.1}$$

Since Θ consists of linear functionals on V_1 continuous in the $\langle \, , \, \rangle_1$-topology, the simultaneous kernel $\ker\Theta$ is a closed subspace of V_1.

Lemma 2. $D \cap \ker\Theta$ *is dense in* V.

Proof. This follows from the general fact that for a continuous inclusion of Hilbert spaces $j : X \longrightarrow Y$, for $D \subset X$ dense in Y, and for a finite-dimensional subspace $\Theta \subset X^*$ such that $j^*(Y^*) \cap \Theta = \{0\}$, we have that $D \cap \ker \Theta \subset X$ is dense in Y.

For completeness, we recall the simple proof. Consider first Θ of dimension 1, spanned by θ. Since $\theta \notin j^*(Y^*)$, θ cannot be continuous in the Y-topology on dense D. This provides for each $\varepsilon > 0$ an element $x_\varepsilon \in D$ with $|x_\varepsilon|_Y < \varepsilon$ and $\theta(x_\varepsilon) = 1$. Given $y \in Y$, density of D in Y yields a sequence z_n in D approaching y in the Y-topology. If $\theta(z_n) = 0$ for infinitely many n, there is nothing to prove. Otherwise, the sequence $z_n' = z_n - \theta(z_n) \cdot x_{2^{-n}/\theta(z_n)}$ is in $\ker \Theta$ and still $z_n' \to y$ in the Y-topology, because

$$|\theta(z_n) \cdot x_{2^{-n}/\theta(z_n)}|_Y < |\theta(z_n)| \cdot \frac{2^{-n}}{|\theta(z_n)|} = 2^{-n} \to 0.$$

Induction on dimension completes the proof. ***Q.E.D.***

Let S_Θ be S restricted to the smaller domain $D_\Theta := D \cap \ker \Theta$ and let $(S_\Theta, D_\Theta)^\sim$ be the Friedrichs extension associated to (S_Θ, D_Θ), with domain $\widetilde{D}_\Theta$, which is indeed dense, by the preceding Lemma 2. Our next goal is the analogue of Lemma 1 for the domain $\widetilde{D}_\Theta$. In order to do this, we need some preparatory observations.

The extension

$$(S_\Theta)^\# \,:\, V_1 \cap \ker \Theta \longrightarrow (V_1 \cap \ker \Theta)^*$$

is defined in the same way as $S^\#$, by

$$(S_\Theta)^\#(x)(y) = \langle x, y \rangle_1 \qquad (x,\, y \in V_1 \cap \ker \Theta).$$

Let

$$t_\Theta : V_1 \cap \ker \Theta \longrightarrow V_1, \quad t_\Theta^* : (V_1)^* \longrightarrow (V_1 \cap \ker \Theta)^*$$

be the inclusion and its adjoint. Assume that Θ is stable under the extension of the conjugation map to V_{-1}.

Lemma 3. $(S_\Theta)^\# = t_\Theta^* \circ S^\# \circ t_\Theta$.

Proof. Lemma 2 shows that D_Θ is dense in V in the V-topology, so formation of the Friedrichs extension as an unbounded self-adjoint operator (densely defined) on V is legitimate. For $x,\, y \in V_1 \cap \ker \Theta$ we have

$$(t_\Theta^* \circ S^\# \circ t_\Theta)(x)(y) = S^\#(x)(y) = \langle t_\Theta x, t_\Theta \overline{y} \rangle_1 = \langle x, \overline{y} \rangle_1 = (S_\Theta)^\#(x)(y),$$

which is the statement of the lemma. *Q.E.D.*

The following re-characterization of the Friedrichs extension of the restriction is straightforward, but essential.

Theorem 4. *The domain $\widetilde{D}_\Theta$ of $\widetilde{S}_\Theta$ is*

$$\begin{aligned}\widetilde{D}_\Theta &= \{x \in V_1 \cap \ker\Theta \,:\, (S^\# \circ t_\Theta)x \in (j^* \circ \Lambda)V + \Theta\}.\\ &= \{x \in V_1 \cap \ker\Theta \,:\, S^\#_\Theta x \in (t^*_\Theta \circ j^* \circ \Lambda)V\}.\end{aligned}$$

We have $\widetilde{S}_\Theta x = y$, with $x \in V_1 \cap \ker\Theta$ and $y \in V$, if and only if $(S^\# \circ t_\Theta)x = (j^* \circ \Lambda)y + \theta$ for some $\theta \in \Theta$.

Proof. The Friedrichs extension $\widetilde{S}_\Theta$ is characterized by

$$\langle z, (\widetilde{S_\Theta})^{-1} y\rangle_1 = \langle z, y\rangle \qquad (z \in D_\Theta,\, y \in V).$$

Given $S^\# x = (j^* \circ \Lambda)y + \theta$ with $x \in V_1 \cap \ker\Theta$, $y \in V$, and $\theta \in \Theta$, take $z \in D_\Theta$ and compute

$$\begin{aligned}\langle x, \overline{z}\rangle_1 &= (S^\# x)(z) = ((j^* \circ \Lambda)y + \theta)(z) = (j^*\overline{y})(z) + \theta(z) = \langle z, \overline{y}\rangle + 0\\ &= \langle y, \widetilde{S_\Theta}^{-1} S\overline{z}\rangle = \langle \widetilde{S_\Theta}^{-1} y, S\overline{z}\rangle = \langle \widetilde{S_\Theta}^{-1} y, \overline{z}\rangle_1,\end{aligned}$$

thus showing that $\widetilde{S}_\Theta^{-1} x = y$. On the other hand, by Lemma 3, $(S_\Theta)^\# x = y$ if and only if $(S^\# \circ t_\Theta)x = y + \theta$ for some $\theta \in \ker t^*_\Theta$, and $\ker t^*_\Theta$ is the closure of Θ in V_{-1}. Since Θ is finite-dimensional, this closure is Θ itself.

The second description of the domain of the Friedrichs extension is immediate from the previous lemma and from the fact that Θ is the kernel of t^*_Θ. *Q.E.D.*

Corollary 5. *With the hypotheses of the theorem, for $x \in V_1$ with $(S^\# - \lambda)x = \theta$ with $\theta \in \Theta$, if $x \in \ker\Theta$ then x is a λ-eigenfunction for the self-adjoint operator $\widetilde{S}_\Theta \geqslant S \geqslant c > 0$. In that case, λ is real and $\lambda \geqslant c > 0$. If S is merely non-negative the same conclusion holds, except for the weaker inequality $\lambda \geqslant 0$.*

Proof. The first part of the corollary is immediate. For the second part, the condition $S \geqslant c > 0$ imposed on S at the beginning of Subsection 2.1 can be removed, replacing it by non-negativity, by applying the first conclusion of the corollary to the operator $S + c$ and noting that the new eigenvalues are obtained by making a shift by c. *Q.E.D.*

In the previous extension-of-restriction scenario, the ambient Hilbert space did not change. Now we consider a scenario in which the ambient Hilbert space changes. Such a situation arose in [20], and was used in [4]. Consider symmetric S with dense domain D on Hilbert space V with conjugation $v \to \overline{v}$, and $\langle Sv, v\rangle \geq \langle v, v\rangle$ for all $v \in D$. Let Θ be an S-stable, conjugation-stable, not necessarily finite-dimensional subspace of D. Let V^Θ be the orthogonal complement of Θ in V, with respect to the hermitian inner product on V. Let $D_\Theta = D \cap V^\Theta$. Let Θ_{-1} be the closure of Θ in V_{-1} in the V_{-1} topology. The relevance of the S-stability of Θ is as expected, namely, that S restricted to D_Θ really does map to V^Θ:

Lemma 6. $S(D_\Theta) \subset V^\Theta$.

Proof. For $v \in D_\Theta$ and $\theta \in \Theta \subset D$,

$$\langle Sv, \theta\rangle = \langle v, S\theta\rangle \in \{\langle v, \theta'\rangle : \theta' \in \Theta\} = \{0\}$$

giving the indicated inclusion. *Q.E.D.*

Unlike the previous situation, we must *assume* that D_Θ is dense in V^Θ, and dense in $V_1 \cap V^\Theta$ in the V_1 topology. We prove the requisite density for cases of interest to us in Lemma 20. Let $(V_1)^\Theta$ be the closure of D_Θ in V_1 with respect to the V_1 topology. Let S_Θ be the restriction of S to D_Θ, and $S^\# : V_1 \to V_{-1}$ by $(S^\# v)(w) = \langle v, \overline{w}\rangle_1$. Let $t_\Theta : V_1^\Theta \to V_1$ be the inclusion, and $t_\Theta^* : V_{-1} \to (V_1^\Theta)^*$ the adjoint. Let $\widetilde{S}_\Theta$ be the Friedrichs extension of S_Θ. Let $(S_\Theta)^\# : V_1^\Theta \to (V_1^\Theta)^*$ by $((S_\Theta)^\# v)(w) = \langle v, \overline{w}\rangle_1$ for $v, w \in V_1^\Theta$. The present analogue of Lemma 3 is

Lemma 7. $(S_\Theta)^\# = t_\Theta^* \circ S^\# \circ t_\Theta$.

Proof. Identical to that of Lemma 3. *Q.E.D.*

The analogue of Theorem 4 has a nearly identical form:

Theorem 8. *The domain of $\widetilde{S}_\Theta$ is*

$$\begin{aligned}\widetilde{D}_\Theta &= \{v \in V_1^\Theta : (S^\# \circ t_\Theta)v \in (j^* \circ \Lambda)V^\Theta + \Theta_{-1}\}\\ &= \{v \in V_1^\Theta : (S_\Theta)^\# v \in (t_\Theta^* \circ j^* \circ \Lambda)V^\Theta\}\end{aligned}$$

We have $\widetilde{S}_\Theta x = y$ for $x \in \widetilde{D}_\Theta$ and $y \in (j^* \circ \Lambda)V^\Theta$ if and only if $(S^\# \circ t_\Theta)x = y + \theta$ for some $\theta \in \Theta_{-1}$.

Proof. The argument is essentially identical to that of Theorem 4. While Θ gives (continuous) functionals on V, via the hermitian inner product on V, the closure

Θ_{-1} in V_{-1} gives (continuous) linear functionals on V_1, via duality. Because the pairing $V_1 \times V_{-1} \to \mathbb{C}$ extends the restriction to $D \times D$ of the $V \times V \to \mathbb{C}$ pairing, $\ker \Theta_{-1} = V_1 \cap V^\Theta$. By assumption, $V^\Theta = V_1 \cap V^\Theta$.

As usual, the extension $\widetilde{S}_\Theta$ is characterized by $\langle z, (\widetilde{S_\Theta})^{-1} y\rangle_1 = \langle z, y\rangle$ for $z \in D_\Theta$ and $y \in V^\Theta$. Given $S^\# x = y + \theta$ with $x \in V_1^\Theta$, $y \in V^\Theta$, and $\theta \in \Theta_{-1}$, take $z \in D_\Theta$ and compute

$$\begin{aligned}\langle x, \overline{z}\rangle_1 &= (S^\# x)(z) = ((j^* \circ \Lambda)y + \theta)(z) = (j^* \overline{y})(z) + \theta(z) = \langle z, \overline{y}\rangle + 0 \\ &= \langle y, \widetilde{S_\Theta}^{-1} S\overline{z}\rangle = \langle \widetilde{S_\Theta}^{-1} y, S\overline{z}\rangle = \langle \widetilde{S_\Theta}^{-1} y, \overline{z}\rangle_1,\end{aligned}$$

Thus, $\widetilde{S}_\Theta^{-1} x = y$. On the other hand, by Lemma 3, $(S_\Theta)^\# x = y$ if and only if $(S^\# \circ t_\Theta)x = y + \theta$ for some $\theta \in \ker t_\Theta^*$, and $\ker t_\Theta^*$ is the closure Θ_{-1} of Θ in V_{-1}. The second description of the domain of the Friedrichs extension is immediate from the previous lemma and from the fact that Θ is the kernel of t_Θ^*. *Q.E.D.*

3 Eisenstein-Sobolev spaces

Let Δ be the hyperbolic Laplacian

$$\Delta = y^2\Big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\Big) \tag{3.1}$$

on the upper half-plane $\mathfrak{H}$, Let $\Gamma = SL_2(\mathbb{Z})$. The standard inner product is

$$\langle f, \overline{g}\rangle = \int_{\Gamma\backslash\mathfrak{H}} f(z)\overline{g(z)}\,\mathrm{d}\omega \tag{3.2}$$

(the Petersson inner product) with respect to the hyperbolic area element $\mathrm{d}\omega_z = y^{-2}\mathrm{d}x\,\mathrm{d}y$. The hyperbolic area of a fundamental domain of $\Gamma\backslash\mathfrak{H}$ is $\langle 1, 1\rangle = \pi/3$.

Let S be $-\Delta$ restricted to have domain

$$\mathfrak{E}_c^\infty \;=\; \Big\{\Psi_\varphi(z) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \varphi(\Im(\gamma z)) \;:\; \varphi \in C_c^\infty(0,\infty)\Big\}. \tag{3.3}$$

These Ψ_φ are *pseudo-Eisenstein series* (the incomplete theta series of other authors) with test-function data φ. The *conjugation map* $f \to \overline{f}$ on $\mathfrak{E}_c^\infty$ is the expected pointwise conjugation. It commutes with Δ and S, and $\mathfrak{E}_c^\infty$ is stable by conjugation. The ambient Hilbert space is the $L^2(\Gamma\backslash\mathfrak{H})$ completion $\mathfrak{E}^0$ of $\mathfrak{E}_c^\infty$. The operator S is a non-negative operator, so the previous discussion of Friedrichs extensions applies to $1+S$, for example, and thereby to S itself, as unbounded operator on $\mathfrak{E}^0$, with domain $\mathfrak{E}_c^\infty$.

3.1 Eisenstein series

Let Γ_∞ be the stabilizer of $i\infty$. The Eisenstein series associated to the cusp at $i\infty$ and $z \in \Gamma\backslash\mathfrak{H}$ is explicitly given when $\Re(s) > 1$ by

$$E_s(z) := \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \Im(\gamma z)^s \; = \; \frac{1}{2}\sum_{(c,d)=1}{}' \; \frac{y^s}{|cz+d|^{2s}} \tag{3.4}$$

and by analytic continuation for general s, where the sum is over coprime integer pairs c, d, and the pair $0, 0$ is also excluded.

The Eisenstein series are functions of the two complex variables z and s, automorphic in z, while s is the *spectral parameter*. The family of functions $z \to E_s(z)$ of z form a meromorphic automorphic-function-valued function of s, with a simple pole at $s=1$ and infinitely many poles for $\Re(s) < \frac{1}{2}$. For s not a pole, $z \to E_s(z)$ is a real-analytic function of the variable $z \in \Gamma\backslash\mathfrak{H}$.

By the $SL_2(\mathbb{R})$-invariance of the Laplacian, $(-\Delta - \lambda_s)E_s = 0$, where $\lambda_s = s(1-s)$. The Eisenstein series satisfy the functional equation $E_s = c_s E_{1-s}$ with

$$c_s = \sqrt{\pi}\,\frac{\Gamma(s-\frac{1}{2})\zeta(2s-1)}{\Gamma(s)\zeta(2s)} = \frac{\xi(2-2s)}{\xi(2s)}, \tag{3.5}$$

where $\xi(s)$ is the completed Riemann zeta function and we have used the functional equation of the zeta function. This also yields $c_s c_{1-s} = 1$.

Lastly, the residue of the simple pole of the Eisenstein series E_s at $s=1$ is the constant $1/\langle 1,1\rangle = 3/\pi$.

3.2 Heegner points and Eisenstein series

It is a well known yet remarkable fact that, for Γ an arithmetic group, values of certain Eisenstein series for $\Gamma\backslash\mathfrak{H}$ have arithmetical significance. Recall that a *fundamental discriminant* is a product of relatively prime factors of the form

$$-4,\; 8,\; -8,\; (-1)^{(p-1)/2}p,$$

where p is an odd prime. Associated to a fundamental discriminant d there is a real, primitive character

$$\chi_d(n) = \left(\frac{d}{n}\right)$$

where (d/n) is the Kronecker symbol, which enjoys the multiplicativity

$$\chi_d(n)\chi_{d'}(n) = \chi_{dd'}(n) \quad \text{for } d \text{ and } d' \text{ coprime.}$$

The absolute value $|d|$ is the *modulus* of the character. The fundamental discriminants are all numbers, positive or negative, of the form N with N square-free and $N \equiv 1 \bmod 4$ or of the form $4N$ with $N \equiv 2$ or $3 \bmod 4$.

From now on d will denote a negative fundamental discriminant. The integral, positive-definite, Lagrange-reduced quadratic forms of discriminant d are

$$Q(x, y) := Ax^2 + Bxy + Cy^2$$

with

$$d = B^2 - 4AC, \;\; |B| \le A \le C, \quad \text{(and when } A = |B| = 1 \text{ then } B = -1).^1$$

The root

$$z_Q = \frac{-B + i\sqrt{|d|}}{2A} \in \Gamma\backslash\mathfrak{H}$$

of the equation $Az^2 + Bz + C = 0$ is the *Heegner point* associated to that reduced quadratic form. We have

$$y_Q^{-1} \cdot |mz_Q + n|^2 = \left(\frac{\sqrt{|d|}}{2}\right)^{-1} Q(n, -m).$$

From this, one shows that for discriminants $d < -4$ the value $E_s(z_Q)$ of the Eisenstein series is

$$E_s(z_Q) = \left(\frac{\sqrt{|d|}}{2}\right)^s \zeta(2s)^{-1}\zeta(s, \mathfrak{z}_Q) \tag{3.6}$$

[1] In this case the two quadratic forms $x^2 \pm xy + \frac{|d|+1}{4}y^2$ are equivalent by $(x, y) \to (x, x \mp y)$ and here we choose the one with $B = -1$ as a representative, the so-called *ambiguous form*.

where $\mathfrak{z}_Q$ is the ideal class of the fractional ideal $[1, z_Q]$ of the imaginary quadratic field $\mathbb{Q}(\sqrt{d})$. [2] Therefore, summing over the $h(d)$ ideal classes we have

$$\sum_{i=1}^{h(d)} E_s(z_{Q_i}) = \left(\frac{\sqrt{|d|}}{2}\right)^s \zeta(2s)^{-1}\zeta(s, \mathbb{Q}(\sqrt{d})) = \left(\frac{\sqrt{|d|}}{2}\right)^s \frac{\zeta(s)}{\zeta(2s)} L(s, \chi_d) \quad (3.7)$$

where z_{Q_i} runs over the $h(d)$ Heegner points for the fundamental discriminant d and where χ_d is the quadratic character associated to $\mathbb{Q}(\sqrt{d})$.

3.3 Spectral decomposition and spectral synthesis

The spectral theory of the Laplacian on $L^2(\Gamma\backslash\mathfrak{H})$, where Γ is a discrete subgroup of $SL_2(\mathbb{R})$ acting on the upper half-plane $\mathfrak{H}$ and of finite covolume, is well understood: see for example [6], or Iwaniec' monograph [18]. This decomposes $L^2(\Gamma\backslash\mathfrak{H})$ as the direct orthogonal sum of the L^2 cuspidal discrete spectrum, constants (L^2 residues of Eisenstein series), and eigen-packets associated to a continuous spectrum generated by the Eisenstein series. Here we only consider $\Gamma = SL_2(\mathbb{Z})$, which has just one family of Eisenstein series giving the continuous spectrum, attached to the single cusp $i\infty$. For $f \in C_c^\infty(\Gamma\backslash\mathfrak{H})$, the spectral expansion is

$$f(z) \;=\; \sum_F \langle f, F\rangle \cdot F(z) + \frac{\langle f, 1\rangle \cdot 1}{\langle 1, 1\rangle} + \frac{1}{4\pi i}\int_{(\frac{1}{2})} \langle f, E_s\rangle \cdot E_s(z)\, ds$$

where F runs over an orthonormal basis for L^2 cuspforms. As usual, $\langle f, E_s\rangle$ cannot be the L^2 pairing, because E_s is not in L^2, but by standard abuse of notation refers to the integral $\int_{\Gamma\backslash\mathfrak{H}} f(z)\, E_{1-s}(z)\, \mathrm{d}\omega_z$, which converges absolutely for automorphic test functions f. For test functions f, the right-hand side converges uniformly pointwise in z. We have Plancherel for test functions:

$$|f|^2_{L^2} \;=\; \sum_F |\langle f, F\rangle|^2 + \frac{|\langle f, 1\rangle|^2}{\langle 1, 1\rangle} + \frac{1}{4\pi i}\int_{(\frac{1}{2})} |\langle f, E_s\rangle|^2\, ds$$

[2] If the discriminant is -3 or -4 the associated quadratic forms $Q(x, y)$ are the ambiguous form $x^2 - xy + y^2$ and $x^2 + y^2$. Besides the obvious automorphism $(x, y) \to (-x, -y)$ arising from $-I \in \Gamma_\infty$ they have the automorphisms $(x, y) \to (x - y, x)$ and $(x, y) \to (y, -x)$ of order 6 and 4, so equations (3.6) and (3.7) must be corrected by factors 3 and 2 in the right-hand side.

Extend the spectral expansion to $f \in L^2(\Gamma\backslash\mathfrak{H})$ by isometry, and write $\mathcal{E}f$ for the extension of $f \to (s \to \langle f, E_s\rangle)$. Pseudo-Eisenstein series $f \in \mathfrak{E}_c^\infty$ with test function data are orthogonal to cuspforms, so for such automorphic forms the spectral expansion and Plancherel become

$$f \;=\; \frac{\langle f,1\rangle\cdot 1}{\langle 1,1\rangle} + \frac{1}{4\pi i}\int_{(\frac{1}{2})} \mathcal{E}f(s)\cdot E_s\, ds$$

and

$$|f|^2_{L^2} \;=\; \frac{|\langle f,1\rangle|^2}{\langle 1,1\rangle} + \frac{1}{4\pi i}\int_{(\frac{1}{2})} |\mathcal{E}f(s)|^2\, ds$$

The operator S, $-\Delta$ restricted to $\mathfrak{E}_c^\infty$, is *symmetric* because the compact support of pseudo-Eisenstein series with test-function data allows integration by parts. For $f \in \mathfrak{E}_c^\infty$ we have the *spectral relation*, intertwining of S and multiplication by λ_s,

$$\begin{aligned}\mathcal{E}(Sf)(s) &= \int_{\Gamma\backslash\mathfrak{H}} (-\Delta)f(z)\cdot E_{1-s}(z)\,\mathrm{d}\omega_z = \int_{\Gamma\backslash\mathfrak{H}} f(z)\cdot(-\Delta)E_{1-s}(z)\,\mathrm{d}\omega_z \\ &= \int_{\Gamma\backslash\mathfrak{H}} f(z)\cdot\lambda_s\, E_{1-s}(z)\,\mathrm{d}\omega_z = \lambda_s\cdot\mathcal{E}f(s).\end{aligned}$$

Thus, for all $0 \leqslant \ell \in \mathbb{Z}$,

$$\mathcal{E}(S^\ell f)(s) = \lambda_s^\ell\cdot\mathcal{E}f(s) \qquad (\text{for } f \in D). \tag{3.8}$$

3.4 Eisenstein-Sobolev spaces $\mathfrak{E}^r$

For $r \in \mathbb{R}$, the r-th global automorphic Sobolev space $H^r(\Gamma\backslash\mathfrak{H})$ is the completion of $C_c^\infty(\Gamma\backslash\mathfrak{H})$ with respect to the r-th Sobolev norm, defined on automorphic test functions f by

$$\begin{aligned}|f|^2_{H^r} \;=\; &\sum_F |\langle f,F\rangle|^2\cdot(1+\lambda_{s_F})^r + \frac{|\langle f,1\rangle|^2\cdot(1+\lambda_1)^r}{\langle 1,1\rangle} \\ &+\frac{1}{4\pi i}\int_{(\frac{1}{2})} |\langle f,E_s\rangle|^2\cdot(1+\lambda_s)^r\, ds\end{aligned}$$

The r-th Eisenstein-Sobolev space is

$$\mathfrak{E}^r = \text{completion of } \mathfrak{E}^\infty_c \text{ with respect to } |\cdot|_{H^r}$$

Let $X^r = \mathbb{C} \oplus X^r_o$ where X^r_o is the weighted L^2-space of measurable functions g on $\frac{1}{2} + i\mathbb{R}$ such that

$$\int_{-\infty}^{\infty} |g(\frac{1}{2}+it)|^2 \cdot (\frac{1}{4}+t^2)^r \, dt \quad < \quad +\infty$$

For all $r \in \mathbb{R}$, Plancherel restricts (for $r \geq 0$) or extends (for $r \leq 0$) to an isometry $\mathfrak{E}^r \to X^r$, by $f \to \frac{\langle f,1\rangle}{\langle 1,1\rangle^{\frac{1}{2}}} \oplus \mathcal{E}f$. Let

$$\mathfrak{E}^\infty = \bigcap_r \mathfrak{E}^r = \lim_r \mathfrak{E}^r \qquad \text{and} \qquad \mathfrak{E}^{-\infty} = \bigcup_r \mathfrak{E}^r = \operatorname*{colim}_r \mathfrak{E}^r$$

in the category of locally convex topological spaces. There is the expected hermitian pairing,

$$\langle f, \theta\rangle_{\mathfrak{E}^r \times \mathfrak{E}^{-r}} = \frac{\langle f,1\rangle \cdot \overline{\langle \theta,1\rangle}}{\langle 1,1\rangle} + \frac{1}{4\pi i}\int_{(\frac{1}{2})} \mathcal{E}f(s) \cdot \overline{\mathcal{E}\theta(s)} \, \mathrm{d}s$$

For all $r \in \mathbb{R}$ the map $S : \mathfrak{E}^\infty_c \to \mathfrak{E}^\infty_c$ is continuous when the domain is given the $\mathfrak{E}^r$ topology and the range is given the $\mathfrak{E}^{r-2}$ topology. Extending by continuity defines L^2*-differentiation* $\mathfrak{E}^r \to \mathfrak{E}^{r-2}$.

3.5 Automorphic Dirac delta distributions

The *pre-trace formula* (as in [18] and elsewhere) is

$$\sum_{F:|\lambda(F)|\leqslant T} |F(z_0)|^2 + \frac{|\langle F,1\rangle|^2}{\langle 1,1\rangle} + \frac{1}{4\pi i}\int_{(\frac{1}{2})} |E_s(z_0)|^2 \, \mathrm{d}s \quad \ll_C \quad T^2$$

for z_0 in a fixed compact subset $C \subset \Gamma\backslash\mathfrak{H}$, where F runs over an orthonormal basis for the cuspidal spectrum and $SF = \lambda(F)\,F$. In particular, dropping the cuspidal part we have

$$\frac{|\langle F,1\rangle|^2}{\langle 1,1\rangle} + \frac{1}{4\pi i}\int_{(\frac{1}{2})} |E_s(z_0)|^2 \, \mathrm{d}s \ll_C T^2$$

Integrating by parts, the function $s \to E_s(z_0)$ is in the weighted L^2 space $X^{-1-\varepsilon}$, so $E_s(z_0) = \mathcal{E}\theta(s)$ for some $\theta \in \mathfrak{E}^{-1-\varepsilon}$, for all $\varepsilon > 0$, and

$$\theta = \frac{\langle \theta, 1\rangle \cdot 1}{\langle 1, 1\rangle} + \frac{1}{4\pi i}\int_{(\frac{1}{2})} E_{1-s}(z_0) \cdot E_s \, \mathrm{d}s \qquad \text{(as an element of } \mathfrak{E}^{-1-\varepsilon}\text{)}$$

Define the non-cuspidal Dirac δ distribution, or Eisenstein-Dirac δ distribution, by

$$\delta^{\mathrm{nc}}_{z_0} := \frac{1}{\langle 1, 1\rangle} + \frac{1}{4\pi i}\int_{(\frac{1}{2})} E_{1-s}(z_0) \cdot E_s \, \mathrm{d}s \qquad \text{(as an element of } \mathfrak{E}^{-1-\varepsilon}\text{)} \tag{3.9}$$

That is, $\delta^{\mathrm{nc}}_{z_o}(s) = E_{1-s}(z_o)$. By design, its action on $f \in \mathfrak{E}^{1+\varepsilon}$ is $\delta^{\mathrm{nc}}_{z_0} f = f(z_0)$. From evaluating the hermitian pairing on $\mathfrak{E}^{1+\varepsilon} \times \mathfrak{E}^{-1-\varepsilon}$:

$$\begin{aligned}
\langle f, \delta^{\mathrm{nc}}_{z_o}\rangle_{\mathfrak{E}^{1+\varepsilon}\times\mathfrak{E}^{-1-\varepsilon}} &= \frac{\langle f, 1\rangle \cdot \overline{\langle \delta^{\mathrm{nc}}_{z_o}, 1\rangle}}{\langle 1, 1\rangle} + \frac{1}{4\pi i}\int_{(\frac{1}{2})} \mathcal{E}f(s) \cdot \overline{\mathcal{E}\delta^{\mathrm{nc}}_{z_o}(s)} \, \mathrm{d}s \\
&= \frac{\langle f, 1\rangle \cdot 1}{\langle 1, 1\rangle} + \frac{1}{4\pi i}\int_{(\frac{1}{2})} \mathcal{E}f(s) \cdot \overline{E_s(z_o)} \, \mathrm{d}s
\end{aligned}$$

At least for f a test-function pseudo-Eisenstein series, this is $f(z_o)$. The estimate is uniform for z_0 in compact subsets of $\Gamma\backslash\mathfrak{H}$, so the map $z_0 \to \delta^{\mathrm{nc}}_{z_0}$ is a continuous $\mathfrak{E}^{-1-\varepsilon}$-valued function of z_0. Thus, for test-function pseudo-Eisenstein series f, by the Cauchy-Schwarz-Bunyakowsky inequality,

$$\begin{aligned}
\sup_{z_o\in C} |f(z_o)| = \sup_{z_o\in C} |\langle f, \delta^{\mathrm{nc}}_{z_o}\rangle| &\le \sup_{z_o\in C} |f|_{\mathfrak{E}^{1+\varepsilon}} \cdot |\delta^{\mathrm{nc}}_{z_o}|_{\mathfrak{E}^{-1-\varepsilon}} \\
&= |f|_{\mathfrak{E}^{1+\varepsilon}} \cdot \sup_{z_o\in C} |\delta^{\mathrm{nc}}_{z_o}|_{\mathfrak{E}^{-1-\varepsilon}} \ll_{C,\varepsilon} |f|_{\mathfrak{E}^{1+\varepsilon}}
\end{aligned}$$

That is, the seminorms obtained by taking suprema on compacta are dominated by the $\mathfrak{E}^{1+\varepsilon}$ norm. This proves the *Sobolev embedding* $\mathfrak{E}^{1+\varepsilon} \subset C^0(\Gamma\backslash\mathfrak{H})$, with the latter topologized by suprema on compact subsets. Thus, $\delta^{\mathrm{nc}}_{z_o}(f) = f(z_o)$ for all $f \in \mathfrak{E}^{1+\varepsilon}$.

As a corollary of the above argument, we again see the expected $\mathcal{E}\delta^{\mathrm{nc}}_{z_o}(s) = E_{1-s}(z_o)$.

3.6 Eisenstein-Heegner distributions

For a fundamental discriminant $d < -4$, let H_d be the set of Heegner points in $\Gamma\backslash\mathfrak{H}$ representing the ideal classes of the ring of integers $\mathbb{Q}(\sqrt{d})$, and let θ_d be the functional

$$\theta_d = \sum_{z \in H_d} \delta_z^{\mathrm{nc}} \in \mathfrak{E}^{-1-\varepsilon} \tag{3.10}$$

for all $\varepsilon > 0$. We call this the *Eisenstein-Heegner distribution* attached to the fundamental discriminant d. The cardinality $h(d) = |H_d|$ is the class number of the quadratic field $\mathbb{Q}(\sqrt{d})$. Let χ_d be the quadratic character attached to $\mathbb{Q}(\sqrt{d})/\mathbb{Q}$. This is a primitive character because d is a fundamental discriminant. For $d < -4$ we have

$$\mathcal{E}\theta_d(s) \;=\; \left(\frac{\sqrt{|d|}}{2}\right)^s \frac{\zeta(s)}{\zeta(2s)} L(s, \chi_d). \tag{3.11}$$

The Eisenstein-Heegner distributions θ_d belong to the space $\mathfrak{E}^{-\frac{1}{2}-\varepsilon}$ for any $\varepsilon > 0$, from the Landau bound $1/\zeta(1+it) = O(\log T)$ and the deeper second moment bound for $\zeta(s)L(s,\chi)$ on the critical line $\Re(s) = \frac{1}{2}$. The $\mathfrak{E}^{-\frac{1}{2}-\varepsilon}$ spectral expansion is (for $d \neq -3$ or -4):

$$\begin{aligned} \theta_d &= \frac{\theta_d(1)\cdot 1}{\langle 1, 1\rangle} + \frac{1}{4\pi i}\int_{(\frac{1}{2})} \theta_d E_{1-s} \cdot E_s \,\mathrm{d}s \\ &= \frac{h(d)}{\langle 1, 1\rangle} + \frac{1}{4\pi i}\int_{(\frac{1}{2})} \left(\frac{\sqrt{|d|}}{2}\right)^{1-s} \frac{\zeta(1-s)L(1-s,\chi_d)}{\zeta(2-2s)} \cdot E_s \,\mathrm{d}s, \end{aligned}$$

3.7 Solving $(-\Delta - \lambda_w)u = \theta$

Let θ be a finite real-linear combination of Eisenstein-Heegner distributions θ_d. For $\Re(w) > \frac{1}{2}$, the equation $(-\Delta - \lambda_w)u = \theta$ has a unique solution $u_{\theta,w}$ in $\mathfrak{E}^{\frac{3}{2}-\varepsilon}$ for every $\varepsilon > 0$, with spectral expansion obtained directly from that of θ via (3.8):

$$u_{\theta,w} = \frac{\theta(1)\cdot 1}{\langle 1, 1\rangle \cdot (\lambda_1 - \lambda_w)} + \frac{1}{4\pi i}\int_{(\frac{1}{2})} \theta E_{1-s} \cdot E_s \frac{\mathrm{d}s}{\lambda_s - \lambda_w}. \tag{3.12}$$

3.8 Constant-term distributions η_a

Let η_a denote the constant term distribution at height $a > 1$ on $f \in \mathfrak{E}_c^\infty$, namely:

$$\eta_a f = \int_0^1 f(x + ia)\,\mathrm{d}x. \tag{3.13}$$

This functional is a compactly-supported, real-valued, regular, Borel measure on $\Gamma\backslash\mathfrak{H}$, so is a continuous functional on $C^0(\Gamma\backslash\mathfrak{H})$. By the remark at the end of 3.5, there is a continuous injection $\mathfrak{E}^{1+\varepsilon} \subset C^0(\Gamma\backslash\mathfrak{H})$ for every $\varepsilon > 0$, so η_a restricts to a continuous functional on $\mathfrak{E}^{1+\varepsilon}$, still denoted η_a. Thus, $\eta_a \in \mathfrak{E}^{-1-\varepsilon}$ for all $\varepsilon > 0$.

As with automorphic Dirac δ and Eisenstein-Dirac δ, we remove a potential ambiguity about correct determination of spectral coefficients $\mathcal{E}\eta_a$. We could again use a variant pre-trace formula, but, instead, we give an argument relevant to subsequent developments:

Proposition 9. *A continuous functional μ on the Fréchet space $C^0(\Gamma\backslash\mathfrak{H})$ given by a compactly-supported real-valued regular Borel measure on $\Gamma\backslash\mathfrak{H}$, restricted to a functional on $\mathfrak{E}^\infty$, is in $\mathfrak{E}^{-1-\varepsilon}$ for every $\varepsilon > 0$, and $\mathcal{E}\mu(s) = \mu(E_{1-s})$.*

Proof. Fix $\varepsilon > 0$. We have a Sobolev imbedding $H^{1+\varepsilon}(\Gamma\backslash\mathfrak{H}) \subset C^o(\Gamma\backslash\mathfrak{H})$ for every $\varepsilon > 0$. The continuous dual of $C^o(\Gamma\backslash\mathfrak{H})$ is exactly compactly-supported regular Borel measures μ. Thus, μ has a natural image in $\mathfrak{E}^{-1-\varepsilon} = (\mathfrak{E}^{1+\varepsilon})^*$, since $\mathfrak{E}^{1+\varepsilon} \subset H^{1+\varepsilon}(\Gamma\backslash\mathfrak{H})$. Thus, there is a spectral expansion in $\mathfrak{E}^{-1-\varepsilon}$:

$$\mu = \frac{\langle \mu, 1\rangle \cdot 1}{\langle 1, 1\rangle} + \frac{1}{4\pi i}\int_{(\frac{1}{2})} \mathcal{E}\mu(s)\cdot E_s \, \mathrm{d}s$$

For any $u \in \mathfrak{E}^{1+\varepsilon}$, on one hand, using $\overline{\mu} = \mu$,

$$\mu(u) = \langle u, \mu\rangle_{\mathfrak{E}^{1+\varepsilon}\times\mathfrak{E}^{-1-\varepsilon}} = \frac{\langle u, 1\rangle \cdot \overline{\langle \mu, 1\rangle}}{\langle 1, 1\rangle} + \frac{1}{4\pi i}\int_{(\frac{1}{2})} \mathcal{E}u(s)\cdot \overline{\mathcal{E}\mu(s)} \, \mathrm{d}s$$

On the other hand, the spectral integral for u converges in $\mathfrak{E}^{1+\varepsilon}$, and is the limit of compactly supported integrals of $C^0(\Gamma\backslash\mathfrak{H})$-valued functions, so

$$\begin{aligned}
\mu(u) &= \mu\Big(\frac{\langle u, 1\rangle \cdot 1}{\langle 1, 1\rangle} + \frac{1}{4\pi i}\int_{(\frac{1}{2})} \mathcal{E}u(s)\cdot E_s \, \mathrm{d}s\Big) \\
&= \mu\Big(\lim_{T\to\infty} \frac{\langle u, 1\rangle \cdot 1}{\langle 1, 1\rangle} + \frac{1}{4\pi i}\int_{|\Im(s)|\le T} \mathcal{E}u(s)\cdot E_s \, \mathrm{d}s\Big) \\
&= \lim_{T\to\infty} \frac{\langle u, 1\rangle \cdot \mu(1)}{\langle 1, 1\rangle} + \frac{1}{4\pi i}\int_{|\Im(s)|\le T} \mathcal{E}u(s)\cdot \mu E_s \, \mathrm{d}s \\
&= \frac{\langle u, 1\rangle \cdot \mu(1)}{\langle 1, 1\rangle} + \frac{1}{4\pi i}\int_{(\frac{1}{2})} \mathcal{E}u(s)\cdot \mu E_s \, \mathrm{d}s
\end{aligned}$$

because the continuous functional μ passes inside the compactly-supported $C^0(\Gamma\backslash\mathfrak{H})$-valued integral of the continuous $C^0(\Gamma\backslash\mathfrak{H})$-valued integrand $s \to \mathcal{E}u(s)\cdot E_s$. (Such standard properties of Gelfand-Pettis vector-valued integrals are recalled in section 6.1.) Since $\overline{\mu} = \mu$, we have $\mu(1) = \overline{\langle \mu, 1\rangle}$, and $\mu E_s = \overline{\mu E_{1-s}}$ for $\Re(s) = \frac{1}{2}$. The two expressions for u agree for all $u \in \mathfrak{E}^{1+\varepsilon}$, giving the proposition. *Q.E.D.*

In fact, $\eta_a \in \mathfrak{E}^{-\frac{1}{2}-\varepsilon}$ for all $\varepsilon > 0$, because the $|\cdot|_{-\frac{1}{2}-\varepsilon}$ norm is

$$|\eta_a|^2_{-\frac{1}{2}-\varepsilon} = \frac{1}{\langle 1,1\rangle} + \frac{1}{4\pi}\int_{-\infty}^{\infty} \frac{|a^s + c_s a^{1-s}|^2}{(1+4t^2)^{\frac{1}{2}+\varepsilon}}\,\mathrm{d}t \ll_a \int_{-\infty}^{\infty} \frac{\mathrm{d}t}{(1+4t^2)^{\frac{1}{2}+\varepsilon}} < \infty.$$

Thus, the $\mathfrak{E}^{-\frac{1}{2}-\varepsilon}$ spectral expansion convergent in $\mathfrak{E}^{-\frac{1}{2}-\varepsilon}$ is

$$\eta_a = \frac{1}{\langle 1,1\rangle} + \frac{1}{4\pi i}\int_{(\frac{1}{2})} (a^{1-s} + c_{1-s}a^s)\,E_s\,\mathrm{d}s \tag{3.14}$$

3.9 Solving $(-\Delta - \lambda_w)u = \eta_a$

For $\Re(w) > \frac{1}{2}$ the equation $(-\Delta - \lambda_w)u = \eta_a$ has an unique solution $v_{w,a} \in \mathfrak{E}^{\frac{3}{2}-\varepsilon}$ for all $\varepsilon > 0$, with spectral expansion obtained directly from that of η_a via (3.8)

$$v_{w,a} = \frac{1}{\langle 1,1\rangle} + \frac{1}{4\pi i}\int_{(\frac{1}{2})} (a^{1-s} + c_{1-s}a^s)\cdot E_s\,\frac{\mathrm{d}s}{\lambda_s - \lambda_w}. \tag{3.15}$$

4 Pseudo-Laplacians on non-cuspidal automorphic spectrum

4.1 Necessary condition for discrete spectrum

For any $\theta \in \mathfrak{E}^{-1}$ with $\overline{\theta} = \theta$ and $\theta \notin \mathfrak{E}^0$, let S_θ be $-\Delta$ restricted to the domain $\mathfrak{E}^\infty_c \cap \ker\theta$. Symmetry of S_θ is inherited from $-\Delta$. The *pseudo-Laplacian* $\widetilde{S}_\theta$ is the Friedrichs extension of S_θ.

As above, for $\theta \in \mathfrak{E}^{-1}$, the distributional equation $(-\Delta - \lambda)u = \theta$ has a unique solution $u \in \mathfrak{E}^1$ for all λ not in $\{0\} \cup [\frac{1}{4}, +\infty)$, via spectral expansions and (3.8).

Theorem 10. *For real $\lambda_w > \frac{1}{4}$, if the equation $(-\Delta - \lambda_w)u = \theta$ has a solution in $\mathfrak{E}^1$ then $\mathcal{E}\theta(w) = 0$. More precisely, existence of a solution implies that, for all $\varepsilon > 0$,*

$$\int_{\Im(w)-\varepsilon}^{\Im(w)+\varepsilon} \Big|\mathcal{E}\theta\Big(\frac{1}{2}+it\Big)\Big| \, \mathrm{d}t \ll_{w,\varepsilon} \varepsilon^{\frac{3}{2}}$$

Proof. The solution u has spectral expansion

$$u = \frac{\langle u, 1\rangle \cdot 1}{\langle 1, 1\rangle} + \frac{1}{4\pi i}\int_{(\frac{1}{2})} \mathcal{E}u(s) \cdot E_s \, \mathrm{d}s \quad (\text{in } \mathfrak{E}^1)$$

and

$$(-\Delta - \lambda_w)u = (\lambda_1 - \lambda_w)\frac{\langle u, 1\rangle \cdot 1}{\langle 1, 1\rangle} + \frac{1}{4\pi i}\int_{(\frac{1}{2})} \mathcal{E}u(s) \cdot (\lambda_s - \lambda_w) \cdot E_s \, \mathrm{d}s \quad (\text{in } \mathfrak{E}^{-1})$$

Since θ itself has a spectral expansion in $\mathfrak{E}^{-1}$, by (3.8) necessarily

$$\mathcal{E}\theta(s) = \mathcal{E}u(s) \cdot (\lambda_s - \lambda_w) \quad (\text{at least as locally-}L^2 \text{ functions on } \tfrac{1}{2} + i\mathbb{R})$$

Further, from the Cauchy-Schwarz-Bunyakowsky inequality,

$$\int_{v_0-\varepsilon}^{v_0+\varepsilon} \Big|\mathcal{E}\theta\Big(\frac{1}{2}+it\Big)\Big|^2 \mathrm{d}t \le \int_{v_0-\varepsilon}^{v_0+\varepsilon} \Big|\mathcal{E}u\Big(\frac{1}{2}+it\Big)\Big| \cdot |t^2 - v_0^2| \, \mathrm{d}t$$

$$\le \Big(\int_{v_0-\varepsilon}^{v_0+\varepsilon} \Big|\mathcal{E}u\Big(\frac{1}{2}+it\Big)\Big| \, \mathrm{d}t\Big)^{\frac{1}{2}} \cdot \Big(\int_{v_0-\varepsilon}^{v_0+\varepsilon} \big|t^2 - v_0^2\big|^2 \, \mathrm{d}t\Big)^{\frac{1}{2}} \ll_{w,\varepsilon} \|\mathcal{E}u\| \cdot \varepsilon^{\frac{3}{2}}$$

as asserted. *Q.E.D.*

As a corollary, we have a *necessary*, but in general not *sufficient*, constraint on possible discrete spectrum of $\widetilde{S}_\theta$:

Corollary 11. *The discrete spectrum $\lambda_w > \frac{1}{4}$ of $\widetilde{S}_\theta$,* **if any**, *is of the form $w(1-w)$ for $w \in \frac{1}{2} + i\mathbb{R}$ such that $\mathcal{E}\theta(w) = 0$, in the sense of the previous theorem.*

Proof. From Theorem 4, any solution $u \in \widetilde{D}_\theta$ to $(\widetilde{S}_\theta - \lambda_w)u = 0$ is a solution to a distributional equation $(-\Delta - \lambda_w)u = c \cdot \theta$ for some constant c. For u not identically 0 and $\lambda_w \neq 0$, the constant c cannot be 0, since $(-\Delta - \lambda_w)u = 0$ has no non-zero solution in $\mathfrak{E}^1$ for $\lambda_w \neq 0$. Thus, without loss of generality, take $c = 1$, and apply the theorem. *Q.E.D.*

Remark 12. *For $\theta \in \mathfrak{E}^{-1+\varepsilon}$ for some $\varepsilon > 0$, theorem 48 will show that on $\Re(w) = \frac{1}{2}$ with $w \neq \frac{1}{2}$, $\mathcal{E}\theta(w) = 0$ is also sufficient for existence of a solution $u \in \mathfrak{E}^1$ to the distributional equation $(-\Delta - \lambda_w)u = \theta$. However, such u is not in the domain of the self-adjoint operator $\widetilde{S}_\theta$ unless also $\theta u = 0$, which does not follow from $\mathcal{E}\theta(w) = 0$ in general.*

Remark 13. *The corollary gives a definite relation between the spectrum of a natural self-adjoint operator and the zeros of $\mathcal{E}\theta(s)$, which, as in section 3.6, in many interesting cases an L-function or a finite linear combination of such. However, there appears to be no general assurance of existence of any discrete spectrum whatsoever.*

Remark 14. *The larger point of our discussion of self-adjoint operators is to prove that various quantities do not vanish in $\Re(w) > \frac{1}{2}$ (off the real line). Unsurprisingly, some non-vanishings are more trivial than might be anticipated. For example, for any $\theta \in \mathfrak{E}^{-1}$, with $u_{\theta,w} \in \mathfrak{E}^1$ a solution of $(-\Delta - \lambda_w)u = \theta$, by spectral theory*

$$\theta u_{\theta,w} = \frac{\langle u, 1\rangle \cdot \overline{\langle \theta, 1\rangle}}{\lambda_1 - \lambda_w} + \frac{1}{4\pi i} \int_{(\frac{1}{2})} \frac{\mathcal{E}\theta(s) \cdot E_s}{\lambda_s - \lambda_w}\, \mathrm{d}s$$

For $\Re(w) > \frac{1}{2}$ and w off $(\frac{1}{2}, 1]$, it is elementary that the imaginary part of that expression is non-zero. That is, $\theta u_{\theta,w} \neq 0$ off the critical line and the real line. That is, although facts about self-adjoint operators do yield these particular conclusions, some of these conclusions are elementary.

4.2 Extensions of restrictions to non-cuspidal pseudo-cuspforms

Here we consider families of restrictions of $-\Delta$ similar to [20], pages 204–206, with attention to details. For fixed $a > 1$, let $\Theta \subset L^2(\Gamma\backslash\mathfrak{H})$ be the space of pseudo-Eisenstein series Ψ_φ with test function φ supported on $[a, \infty)$. Since $\Delta\Psi_\varphi = \Psi_{\Delta\varphi}$ the space Θ is stable under Δ. Let $\mathfrak{E}^0_\Theta$ be the orthogonal complement to Θ in $\mathfrak{E}^0$.

Let S_Θ be the restriction of $-\Delta$ to $\mathfrak{E}^\infty_c \cap \mathfrak{E}^0_\Theta$, and $\widetilde{S}_\Theta$ its Friedrichs extension. To avoid potential ambiguities, we should be sure that S_Θ is densely defined on $\mathfrak{E}^0_\Theta$:

Lemma 15. *$\mathfrak{E}^\infty_c \cap \mathfrak{E}^0_\Theta$ is dense in $\mathfrak{E}^0_\Theta$, and $\mathfrak{E}^\infty_c \cap \mathfrak{E}^0_\Theta$ is dense in $\mathfrak{E}^1 \cap \mathfrak{E}^0_\Theta$ with the $\mathfrak{E}^1$ topology.*

Proof. To show that $\mathfrak{E}^\infty_c \cap \mathfrak{E}^0_\Theta$ is dense in $\mathfrak{E}^0_\Theta$, given a sequence of pseudo-Eisenstein series $\Psi_{\varphi_i} \in \mathfrak{E}^\infty_c$ converging to $f \in \mathfrak{E}^0_\Theta$, we produce a sequence of pseudo-Eisenstein series in $\mathfrak{E}^\infty_c \cap \mathfrak{E}^0_\Theta$ converging to f. We will do so by cutting off the constant terms of the Ψ_{φ_i} at height a. Since the limit f of the Ψ_{φ_i} has constant term vanishing above height $y = a$, that part of the constant terms of the Ψ_{φ_i} must become small. The explicit details are routine. *Q.E.D.*

Essentially as in [20] but restricting to the orthogonal complement $\mathfrak{E}^0$ to cuspforms, we have

Theorem 16. *The resolvent* $(\widetilde{S}_\Theta - \lambda_w)^{-1}$ *is compact for* $\lambda_w \notin \mathbb{R}$*, and has a meromorphic continuation to* $w \in \mathbb{C}$*, giving a compact operator for* w *off a discrete subset of* $(\frac{1}{2} + i\mathbb{R}) \cup [0,1]$*. In particular,* $\widetilde{S}_\Theta$ *has purely discrete spectrum.*

Proof. Since $(1+\widetilde{S}_\Theta)^{-1} : \mathfrak{E}^0_\Theta \to \mathfrak{E}^1_\Theta$ is continuous with the (finer) $\mathfrak{E}^1$-topology on $\mathfrak{E}^1$, we see below that it suffices to demonstrate a Rellich-lemma-type compactness, namely, that the inclusion $\mathfrak{E}^1_\Theta \to \mathfrak{E}^0_\Theta$ is a compact linear map. The corresponding compactness for compact Riemannian manifolds, possibly with boundary, is standard.

The total boundedness criterion for relative compactness requires that, given $\varepsilon > 0$, the image of the unit ball $B \subset \mathfrak{E}^1_\Theta$ by the inclusion into $\mathfrak{E}^0_\Theta$ can be covered by finitely-many balls of radius ε. The Rellich lemma on compact Riemannian manifolds reduces the issue to an estimate on the a-tail of the quotient $\Gamma\backslash\mathfrak{H}$, that is, the image in $\Gamma\backslash\mathfrak{H}$ of $\{z \in \mathfrak{H} : \Im(z) \geq a\}$. Then the necessary estimate on the a-tail will follows from the $\mathfrak{E}^1_\Theta$- boundedness.

We prove that, given $\varepsilon > 0$, there is c sufficiently large so that $\varphi_\infty \cdot B$ lies in a single ball of radius ε inside $L^2(\Gamma\backslash\mathfrak{H})$, that is,

$$\lim_{c\to\infty} \int_{y>c} |f(z)|^2 \, \frac{\mathrm{d}x \, \mathrm{d}y}{y^2} \;\longrightarrow\; 0$$

uniformly for $|f|_{\mathfrak{E}^1} \leq 1$.

A precise choice of smooth truncations and understanding of their $\mathfrak{E}^1$ norms is needed. Fix a smooth real-valued function ψ on $\mathbb{R}$ with $\psi(y) = 0$ for $y \leq 0$, $0 \leq \psi(y) \leq 1$ for $0 < y < 1$, and $\psi(y) = 1$ for $y \geq 1$. For $t > 1$, let $\psi_t(y) = \psi(\frac{y}{t} - 1)$, and form a pseudo-Eisenstein series Ψ_{ψ_t}, a locally finite sum. Then $\Psi_{\psi_t} \cdot f(x+iy)$ is a smoothly cut-off tail of f starting gradually at height t: $(1 - \Psi_{\psi_t}) \cdot f$ is identically 0 in the region where $y \geq 2t$, and in all images of this region under $SL_2(\mathbb{Z})$.

Lemma 17. *The smooth truncation* $\Psi_{\psi_t} \cdot f$ *has* $\mathfrak{E}^1$*-norm dominated by that of* f *itself, with implied constant uniform in* $t \geq 2$ *and in* $f \in \mathfrak{E}^1$.

Proof. Routine computation. *Q.E.D.*

Returning to the proof of Theorem 16, given $c > 1$, we can assume that f has been smoothly truncated so that in the fundamental domain it is supported inside the region where $y \geq c$, and increasing its $\mathfrak{E}^1$ norm by at most a uniform factor. Let the Fourier coefficients of $f(x+iy)$ in x be $\widehat{f}(n)$, functions of y. Take $y \geq c > a$, so the $\widehat{f}(0)$ vanishes. Using Plancherel for the Fourier expansion in x, integrating over the part of Y_∞ above $y = c$, letting $\mathcal{F}$ denote Fourier transform in x, there is a direct computation

$$\begin{aligned}
&\int\int_{y>c} |f|^2 \, \frac{\mathrm{d}x \, \mathrm{d}y}{y^2} \;\leq\; \frac{1}{c^2}\int\int_{y>c} |f|^2 \, \mathrm{d}x \, \mathrm{d}y = \frac{1}{c^2}\sum_{n\neq 0}\int_{y>c} |\widehat{f}(n)|^2 \, \mathrm{d}y \\
&\leq\; \frac{1}{c^2}\sum_{n\neq 0}(2\pi n)^2 \int_{y>c} |\widehat{f}(n)|^2 \, \mathrm{d}y = \frac{1}{c^2}\sum_{n\neq 0}\int_{y>c} \Big| \mathcal{F}\,\frac{\partial f}{\partial x}(n)\Big|^2 \, \mathrm{d}y \\
&= \frac{1}{c^2}\int\int_{y>c} \Big|\frac{\partial f}{\partial x}\Big|^2 \, \mathrm{d}x \, \mathrm{d}y = \frac{1}{c^2}\int\int_{y>c} -\frac{\partial^2 f}{\partial x^2}\cdot \overline{f}(x) \, \mathrm{d}x \, \mathrm{d}y \\
&\leq \frac{1}{c^2}\int\int_{y>c}\Big(-\frac{\partial^2 f}{\partial x^2}\cdot \overline{f}(x) - \frac{\partial^2 f}{\partial y^2}\cdot \overline{f}(x)\Big) \, \mathrm{d}x \, \mathrm{d}y = \frac{1}{c^2}\int\int_{y>c} -\Delta f \cdot \overline{f} \, \frac{\mathrm{d}x \, \mathrm{d}y}{y^2} \\
&\leq \frac{1}{c^2}\int\int_{\Gamma\backslash\mathfrak{H}} -\Delta f \cdot \overline{f} \, \frac{\mathrm{d}x \, \mathrm{d}y}{y^2} = \frac{1}{c^2}\,|f|^2_{\mathfrak{E}^1} \;\leq\; \frac{1}{c^2}
\end{aligned}$$

This uniform bound completes the proof that the image of the unit ball of $\mathfrak{E}^1_\Theta$ in $\mathfrak{E}^0_\Theta$ is totally bounded. Thus, the inclusion is a compact map.

As earlier, Friedrichs' construction shows that $(\widetilde{S}_\Theta - \lambda)^{-1} : \mathfrak{E}^0_\Theta \to \mathfrak{E}^1_\Theta$ is continuous even with the stronger topology of $\mathfrak{E}^1_\Theta$. Thus, the composition $\mathfrak{E}^0_\Theta \to \mathfrak{E}^1_\Theta \subset \mathfrak{E}^0_\Theta$ by

$$f \longrightarrow (\widetilde{S}_\Theta - \lambda)^{-1} f \to (\widetilde{S}_\Theta - \lambda)^{-1} f$$

is the composition of a continuous operator with a compact operator, so is compact. Thus, $(\widetilde{S}_\Theta - \lambda)^{-1} : \mathfrak{E}^0_\Theta \to \mathfrak{E}^0_\Theta$ is a compact operator. Thus, for λ off a discrete set of points in $\mathbb{C}$, $\widetilde{S}_\Theta$ has compact resolvent $(\widetilde{S}_\Theta - \lambda)^{-1}$, and the parametrized family of compact operators $(\widetilde{S}_\Theta - \lambda)^{-1} : \mathfrak{E}^0_\Theta \longrightarrow \mathfrak{E}^0_\Theta$ is meromorphic in $\lambda \in \mathbb{C}$.

We recall the standard argument (see [19], page 187 and preceding, for example) for the fact that that, for a (not necessarily bounded) normal operator T, if T^{-1} exists and is compact, then $(T-\lambda)^{-1}$ exists and is a compact operator for λ off a discrete set in $\mathbb{C}$, and is *meromorphic* in λ. First, from the spectral theory of normal compact operators, the non-zero spectrum of compact T^{-1} is all point spectrum. We claim that the spectrum of T and non-zero spectrum of T^{-1} are in the obvious bijection $\lambda \leftrightarrow \lambda^{-1}$. From the algebraic identities $T^{-1} - \lambda^{-1} = T^{-1}(\lambda - T)\lambda^{-1}$ and $T - \lambda = T(\lambda^{-1} - T^{-1})\lambda$, failure of either $T - \lambda$ or $T^{-1} - \lambda^{-1}$ to be *injective* forces the failure of the other, so the point spectra are identical. For (non-zero) λ^{-1} not an eigenvalue of *compact* T^{-1}, $T^{-1} - \lambda^{-1}$ is injective *and* has a continuous, everywhere-defined inverse. That $S - \lambda$ is *surjective* for compact normal S and $\lambda \neq 0$ not an eigenvalue is an easy part of Fredholm theory. For such λ, inverting the relation $T - \lambda = T(\lambda^{-1} - T^{-1})\lambda$ gives

$$(T - \lambda)^{-1} = \lambda^{-1}(\lambda^{-1} - T^{-1})^{-1}T^{-1}$$

from which $(T - \lambda)^{-1}$ is continuous and everywhere-defined. That is, λ is *not* in the spectrum of T. Finally, $\lambda = 0$ is not in the spectrum of T, because T^{-1} exists and is continuous. This establishes the bijection. *Q.E.D.*

Essentially as in [20], identification of the eigenfunctions for $\widetilde{S}_\Theta$ depends on the cut-off height $a > 1$ and reduction theory. The *truncation* $\wedge^a E_s$ of an Eisenstein series at height $\Im(z) = a$ is as follows. With $y = \Im(z)$, let $\tau_s(z)$ be $y^s + c_s y^{1-s}$ for $y \geq a$ and 0 for $0 < y < a$, and form a pseudo-Eisenstein series

$$\Psi_{s,a}(z) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \tau_s(\gamma z)$$

Even though τ_s is *not* a test function, by reduction theory this is a locally finite sum, so converges uniformly absolutely pointwise. For $a > 1$, inside the standard fundamental domain for $SL_2(\mathbb{Z})$, by reduction theory, $\Psi_{s,a}(z)$ is 0 unless $y \geq a$, and is $y^s + c_s y^{1-s}$ for $y \geq a$. The truncated Eisenstein series is

$$\wedge^a E_s = E_s - \Psi_{s,a}$$

By design, for $a > 1$, inside the standard fundamental domain for $SL_2(\mathbb{Z})$, this truncation makes the constant term vanish above height $y = a$.

Theorem 18. *For $a > 1$, spectral parameters w for eigenvalues $\lambda_w > \frac{1}{4}$ of $\widetilde{S}_\Theta$ are exactly the zeros of the constant term $a^w + c_w a^{1-w}$ of the Eisenstein series E_w with $\Re(w) = \frac{1}{2}$. The corresponding eigenvalues λ_w are simple, the corresponding eigenfunctions are solutions $u \in \mathfrak{E}^1_\Theta$ of the equation $(-\Delta - \lambda)u = \eta_a$, and up to constants these are truncated Eisenstein series $\wedge^a E_w$. Specifically,*

$$(-\Delta - \lambda_w) \wedge^a E_w = 2(1 - 2w)a^{w+1} \cdot \eta_a$$

Remark 19. *In particular, all eigenfunctions with eigenvalues* $\lambda > \frac{1}{4}$ *fail to be smooth, since their constant terms will be continuous, but have discontinuous first derivative (in* $y = \Im(z)$*) at height* a*. In this regard, such eigenfunctions are exotic.*

Proof. Because the homogeneous equation $(-\Delta - \lambda)u = 0$ has no non-zero solution in $\mathfrak{E}^1_\Theta$, it suffices to identify the possible θ in the $\mathfrak{E}^{-1}$ closure Θ_{-1} of Θ that could fit into an equation $(-\Delta-\lambda)u = \theta$ with u in the $\mathfrak{E}^1$ closure of $\mathfrak{E}^\infty_c \cap \mathfrak{E}^1_\Theta$. On one hand, because $a > 1$, Θ_{-1} consists of distributions which, on the standard fundamental domain, have support inside the Siegel set $\mathfrak{S}_a = \{x + iy \in \mathfrak{H} : y \geq a\}$. Further, on $C_a = \Gamma_\infty \backslash \mathfrak{S}_a$, the circle $S^1 = \mathbb{Z}\backslash\mathbb{R}$ acts by translations, descending to the quotient from $\mathfrak{H}$. By reduction theory, the restrictions to C_a of every pseudo-Eisenstein series Ψ_φ with $\varphi \in C^\infty_c[a, \infty)$ are invariant under S^1, so anything in the $\mathfrak{E}^{-1}$ closure is likewise translation-invariant.

On the other hand, $\mathfrak{E}^\infty_c \cap \mathfrak{E}^1_\Theta$ consists of functions with constant term vanishing in $y \geq a$, and taking $\mathfrak{E}^1$ completion preserves this property. Since Θ_{-1} is S^1-invariant and the Laplacian commutes with S^1, it suffices to look at S^1-integral averages of v restricted to some cylinder C_b with $a > b > 1$. Such an integral is a restriction of the constant term $c_P v$ to C_b, and vanishes in $y > a$.

Thus, in the standard fundamental domain, the support of distributions $\theta \in \Theta_{-1}$ fitting into an equation $(-\Delta - \lambda)u = \theta$, with u in the $\mathfrak{E}^1$ closure of $\mathfrak{E}^\infty_c \cap \mathfrak{E}^1_\Theta$, is inside the image of the set where $y = a$, and θ is S^1-invariant. By the classification of distributions supported on submanifolds, such θ is a derivative normal to the circle $\Gamma_\infty\backslash\{x+iy : y = a\}$ followed by application of an S^1-invariant distribution on the circle. The latter must be integration over the circle, by classification of invariant distributions. By standard Sobolev theory, there can be no actual derivatives, for the resulting distribution to lie in the -1 Sobolev space. Thus, up to a constant multiple, θ is the evaluation of constant term at height a functional η_a, namely, $\eta_a f = c_P f(ia)$.

Since $|a^s + c_s a^{1-s}| \leq 2\sqrt{a}$ for $\Re(s) = \frac{1}{2}$, in fact $\eta_a \in \mathfrak{E}^{-\frac{1}{2}-\varepsilon}$ for every $\varepsilon > 0$.

As in Theorem 10 and its proof, from the spectral relation (3.8), for $v \in \mathfrak{E}^1$ with $(-\Delta - \lambda_w)u = \eta_a$, the spectral expansions

$$u = \frac{\langle u, 1\rangle \cdot 1}{\langle 1, 1\rangle} + \frac{1}{4\pi i}\int_{(\frac{1}{2})} \mathcal{E}u(s) \cdot E_s \, \mathrm{d}s \tag{4.1}$$

$$\eta_a = \frac{\langle \eta_a, 1\rangle \cdot 1}{\langle 1, 1\rangle} + \frac{1}{4\pi i}\int_{(\frac{1}{2})} \mathcal{E}\eta_a(s) \cdot E_s \, \mathrm{d}s \tag{4.2}$$

converging in $\mathfrak{E}^1$ and $\mathfrak{E}^{-1}$, respectively, have $\mathcal{E}\eta_a(s) = \eta_a(E_{1-s}) = a^{1-s} + c_{1-s}a^s$, and the expansions are related by $(\lambda_s - \lambda_w) \cdot \mathcal{E}u(s) = \mathcal{E}\eta_a(s)$, so $\mathcal{E}\eta_a(w) = 0$ in a strong sense. Since $\Re(w) = \frac{1}{2}$, by $c_w c_{1-w} = 1$, we have $a^w + c_w a^{1-w} = 0$.

To finish the proof, it suffices to show that $(-\Delta - \lambda_w) \wedge^a E_w$ is a scalar multiple of η_a when $a^w + c_w a^{1-w} = 0$. Indeed, with $a > 1$, in a fundamental domain, away from $y = a$ we have $(-\Delta - \lambda_w) \wedge^a E_w = 0$ locally. Further, in $y > 1$, the differential operator annihilates all Fourier components of E_w but the constant term, and in both $1 < y < a$ and $y > a$ does also annihilate the constant term. To compute near $y = a$, let H be the Heaviside function $H(y) = 0$ for $y < 0$ and $H(y) = 1$ for $y > 0$. Thus, near $y = a$, as functions of y independent of x,

$$
\begin{aligned}
(-\Delta - \lambda_w) \wedge^a E_w &= (-\Delta - \lambda_w)\Big(H(a-y) \cdot (y^w + c_w y^{1-w})\Big) \\
&= (-y^2 \frac{\partial^2}{\partial y^2} - w(1-w))\Big(H(a-y) \cdot (y^w + c_w y^{1-w})\Big) \\
= -y^2\Big(H''(a-y)(y^w + c_w y^{1-w}) &+ 2H'(a-y)(y^w + c_w y^{1-w})' \\
&+ H(a-y)(y^w + c_w y^{1-w})''\Big) - w(1-w)H(a-y)(y^w + c_w y^{1-w}) \\
= -y^2\Big(\delta_a' \cdot (y^w + c_w y^{1-w}) &- 2\delta_a \cdot (w y^{w-1} + (1-w)c_w y^{-w})\Big)
\end{aligned}
$$

Since $a^w + c_w a^{1-w} = 0$, the term with δ_a' vanishes, and the rest simplifies to

$$
\begin{aligned}
(-\Delta - \lambda_w) \wedge^a E_w &= -2a\delta_a \cdot (w a^w + (1-w)c_w a^{1-w}) \\
&= -2\delta_a \cdot (2w-1)a^{w+1}
\end{aligned}
$$

on functions of y independent of x. Thus, this is $2(1-2w)a^{w+1} \cdot \eta_a$. *Q.E.D.*

4.3 Exotic eigenfunction expansions

Keep $a > 1$ fixed, and, as above, Θ the collection of pseudo-Eisenstein series formed from test function data φ supported in $[a, +\infty)$. Any $u \in \mathfrak{E}^1$ with $\eta_b u = 0$ for all $b \geq a > 1$ lies in $\mathfrak{E}^0_\Theta$, so admits a spectral expansion in terms of the eigenfunctions for $\widetilde{S}_\Theta$, converging in the topology of $\mathfrak{E}^0$. However, we want to apply functionals in $\mathfrak{E}^{-1}$ (termwise) to such an spectral expansion, which requires that the expansion converge in $\mathfrak{E}^1$. Thus, we need the following stronger analogue of Lemma 15:

Theorem 20. *With $a > 1$, $\mathfrak{E}_c^\infty \cap \mathfrak{E}_\Theta^0$ is dense in $\mathfrak{E}^1 \cap \mathfrak{E}_\Theta^0$ with the $\mathfrak{E}^1$ topology.*

Proof. Given a sequence of pseudo-Eisenstein series $\Psi_{\varphi_i} \in \mathfrak{E}_c^\infty$ converging to $f \in \mathfrak{E}_\Theta^1$ in the topology of $\mathfrak{E}^1$, we produce a sequence of pseudo-Eisenstein series in $\mathfrak{E}_c^\infty \cap \mathfrak{E}_\Theta^0$ converging to f in the topology of $\mathfrak{E}^1$. We will do so by smooth cut-offs of the constant terms of the Ψ_{φ_i}. Since the limit f of the Ψ_{φ_i} has constant term vanishing above height $y = a$ and is in $\mathfrak{E}_\Theta^0$, that part of the constant terms of the Ψ_{φ_i} becomes small. More precisely, we proceed as follows.

Let g be a smooth real-valued function on $\mathbb{R}$ with $g(y) = 0$ for $y < -1$, $0 \leq g(y) \leq 1$ for $-1 \leq y \leq 0$, and $g(y) = 1$ for $y \geq 0$. For $\varepsilon > 0$, let $g_\varepsilon(y) = g((y-a)/\varepsilon)$. Fix real b with $a > b > 1$. Given $\Psi_{\varphi_i} \to f \in \mathfrak{E}_\Theta^0$, the b-*tail* of the constant term of Ψ_{φ_i} is $\tau_i(y) = c_p\Psi_{\varphi_i}(y)$ for $y \geq b$, and $\tau_i(y) = 0$ for $0 < y \leq b$. By design, $\Psi_{\varphi_i} - \Psi_{g_\varepsilon \cdot \tau_i} \in \mathfrak{E}_c^\infty \cap \mathfrak{E}_\Theta^0$ for small ε. We will show that, as $i \to +\infty$, for ε_i sufficiently small depending on i, the $\mathfrak{E}^1$-norms of $\Psi_{g_{\varepsilon_i} \cdot \tau_i}$ go to 0, so $\Psi_{\varphi_i} - \Psi_{g_{\varepsilon_i} \cdot \tau_i} \to f$ in the $\mathfrak{E}^1$ topology.

For $b > 1$, let $\mathfrak{S}_b = \{x + iy \in \mathfrak{H} : y \geq b,\ |x| \leq \frac{1}{2}\}$. By reduction theory, the cylinder $C_b = \Gamma_\infty \backslash (\Gamma_\infty \cdot \mathfrak{S}_b)$ maps homeomorphically to its image in $\Gamma\backslash\mathfrak{H}$. For $f \in C_c^\infty(\Gamma\backslash\mathfrak{H})$, let

$$|f|^2_{H^1(C_b)} = \int_{C_b} |f(z)|^2 - \Delta f \cdot \overline{f}\ \frac{\mathrm{d}x\,\mathrm{d}y}{y^2} \leq \int_{\Gamma\backslash\mathfrak{H}} |f(z)|^2 - \Delta f \cdot \overline{f}\ \frac{\mathrm{d}x\,\mathrm{d}y}{y^2}$$

For each $b > 1$, let $H^1(C_b)$ be the completion of $C_c^\infty(\Gamma\backslash\mathfrak{H})$ with respect to the semi-norm $|\cdot|_{H^1(C_b)}$, allowing for collapsing. The cylinders C_b admit natural actions of the circle group $S^1 = \mathbb{Z}\backslash\mathbb{R}$, by translation, inherited from the translation of the real part of $x + iy \in \mathfrak{H}$. As usual, this induces a continuous action of S^1 on $H^1(C_b)$. Thus, the map $F \to c_P F$ gives continuous maps of the spaces $H^1(C_b)$ to themselves. Thus, $c_P\Psi_{\varphi_i}$ goes to $c_P f$ in $H^1(C_b)$, and $c_P\Psi_{\varphi_i} \to c_P f = 0$ in $H^1(C_a)$.

To have a useful Leibniz rule for differentiation, it is convenient to rewrite the norms: for $f \in C_c^\infty(\Gamma\backslash\mathfrak{H})$, put

$$|f|^2_{H_1} = |f|^2_{L^2(\Gamma\backslash\mathfrak{H})} + |(|\nabla f|_{\mathfrak{s}})|^2_{L^2(\Gamma\backslash\mathfrak{H})}$$

where ∇ is the left $SL_2(\mathbb{R})$-invariant, right $SO_2(\mathbb{R})$-equivariant tangent-space-valued gradient on $SL_2(\mathbb{R})$, which therefore descends to $\mathfrak{H}$ and to $\Gamma\backslash\mathfrak{H}$, and $|\cdot|_{\mathfrak{s}}$ is a natural $SO_2(\mathbb{R})$-invariant norm on the tangent space(s). More explicitly, let $\mathfrak{s}$ be the space of symmetric 2-by-2 matrices of trace 0, identified with the tangent space at every point of $\mathfrak{H}$ via left translation of the exponential map: for $\beta \in \mathfrak{s}$, as usual the associated left $SL_2(\mathbb{R})$-invariant differential operator X_β is

$$(X_\beta f)(g) = \frac{\partial}{\partial t}\Big|_{t=0} f(ge^{t\cdot\beta})$$

It is easy to describe ∇ in coordinates, even though it is provably independent of coordinates: let $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and put

$$\nabla F(g) = X_h F(g) \cdot h + X_\sigma F(g) \cdot \sigma \ \in \ \mathfrak{s} \otimes_{\mathbb{R}} \mathbb{C}$$

Up to a scalar, the $SO_2(\mathbb{R})$-invariant hermitian inner product $\langle , \rangle_{\mathfrak{s}}$ on the complexified $\mathfrak{s}$, makes h, σ an orthonormal basis for $\mathfrak{s}$. Let $|\cdot|_{\mathfrak{s}}$ be the associated norm. The essential property is the integration by parts identity

$$\int_{\Gamma\backslash\mathfrak{H}} \langle \nabla F_1, \nabla F_2 \rangle_{\mathfrak{s}} = \int_{\Gamma\backslash\mathfrak{H}} -\Delta F_1 \cdot \overline{F}_2$$

for $F_1, F_2 \in C_c^\infty(\Gamma\backslash\mathfrak{H})$. The advantage of this formulation is that, extending ∇ by continuity in the H^1 topology, ∇F exists (in an L^2 sense) for $F \in H^1(C_b)$. Thus, we can say that

$$|F|^2_{H^1(C_b)} = |F|^2_{L^2(\Gamma\backslash\mathfrak{H})} + |\,|\nabla F|_{\mathfrak{s}}|^2_{L^2(\Gamma\backslash\mathfrak{H})}$$

Then

$$\begin{aligned} &|\Psi_{g_\varepsilon\cdot\tau_i}|_{\mathfrak{E}^1} = |\Psi_{g_\varepsilon\cdot\tau_i}|_{H^1(C_{a-\varepsilon})} = |g_\varepsilon \cdot \tau_i|_{H^1(C_{a-\varepsilon})} \\ &\le |(g_\varepsilon - 1)\cdot\tau_i|_{H^1(C_{a-\varepsilon})} + |\tau_i - c_P f|_{H^1(C_{a-\varepsilon})} + |c_P f|_{H^1(C_{a-\varepsilon})} \end{aligned}$$

The middle summand goes to 0:

$$|\tau_i - c_P f|_{H^1(C_{a-\varepsilon})} \le |c_P \Psi_{\varphi_i} - c_P f|_{\mathfrak{E}^1} \le |\Psi_{\varphi_i} - f|_{\mathfrak{E}^1} \longrightarrow 0$$

The first and third summands require somewhat more care. Estimate

$$|(g_\varepsilon - 1)\cdot\tau_i|^2_{H^1(C_{a-\varepsilon})} = \int_{C_{a-\varepsilon}} |(g_\varepsilon - 1)\tau_i|^2 + |\nabla(g_t - 1)\tau_i|^2_{\mathfrak{s}}$$

$$\le \int\limits_{C_{a-\varepsilon}} |g_\varepsilon - 1|^2 \cdot (|\tau_i|^2 + |\nabla\tau_i|^2_{\mathfrak{s}}) + \int\limits_{C_{a-\varepsilon}} |\nabla g_\varepsilon|^2_{\mathfrak{s}} \cdot |\tau_i|^2 + \int\limits_{C_{a-\varepsilon}} 2|g_\varepsilon| \cdot |\nabla g_\varepsilon|_{\mathfrak{s}} \cdot |\tau_i| \cdot |\nabla\tau_i|_{\mathfrak{s}}$$

The first summand in the latter expression goes to 0 as $\varepsilon \to 0^+$ because $g_\varepsilon - 1 = 0$ when $y \geq a$, and τ_i and $|\nabla \tau_i|_{\mathfrak{s}}$ are continuous.

In terms of the coordinates $z = x + iy$ on $\mathfrak{H}$, for a smooth function F a standard computation gives

$$\nabla F = y \frac{\partial F}{\partial x} \cdot \sigma + y \frac{\partial F}{\partial y} \cdot h$$

so

$$|\nabla g_\varepsilon(x+iy)|_{\mathfrak{s}} = |\frac{1}{\varepsilon} \cdot y\, g'((y-a)/\varepsilon) \cdot h|_{\mathfrak{s}} = \frac{1}{\varepsilon} \cdot |y\, g'((y-a)/\varepsilon)| \ll_g \frac{1}{\varepsilon}$$

Similarly, since τ_i is a function of y independent of z, $\nabla \tau_i = y \tau_i'(y) \cdot h$. By the fundamental theorem of calculus and the Cauchy-Schwarz-Bunyakowsky inequality, we recover an easy instance of a Sobolev inequality:

$$\begin{aligned} |\tau_i(a-v)| &= \Big|0 - \int_0^v \tau_i'(a-v)\, \mathrm{d}v\Big| \\ &\leq \Big(\int_0^v |\tau_i'(a-v)|^2\, \mathrm{d}v\Big)^{\frac{1}{2}} \cdot \Big(\int_0^v 1^2\, \mathrm{d}v\Big)^{\frac{1}{2}} = o(\sqrt{v}) \end{aligned}$$

with Landau's little-o notation, since τ_i' is locally L^2. Thus,

$$\begin{aligned} &\int_{C_{a-\varepsilon}} |g_\varepsilon| \cdot |\nabla g_\varepsilon|_{\mathfrak{s}} \cdot |\tau_i| \cdot |\nabla \tau_i|_{\mathfrak{s}} \leq \frac{1}{\varepsilon} \cdot o(1) \cdot \sqrt{\varepsilon} \cdot \int_0^\varepsilon |\nabla \tau_i|_{\mathfrak{s}} \\ &\leq \frac{1}{\varepsilon} \cdot o(1) \cdot \sqrt{\varepsilon} \cdot \Big(\int_0^\varepsilon |\tau_i'|^2\Big)^{\frac{1}{2}} \cdot \Big(\int_0^\varepsilon 1^2\Big)^{\frac{1}{2}} \ll_{\tau_i} \frac{1}{\varepsilon} \cdot o(1) \cdot \sqrt{\varepsilon} \cdot \sqrt{\varepsilon} = o(1) \end{aligned}$$

That is, the summand $\int_{C_{a-\varepsilon}} |g_\varepsilon| \cdot |\nabla g_\varepsilon|_{\mathfrak{s}} \cdot |\tau_i| \cdot |\nabla \tau_i|_{\mathfrak{s}}$ goes to 0. Using the same subordinate estimates,

$$\int_{C_{a-\varepsilon}} |\nabla g_\varepsilon|_{\mathfrak{s}}^2 \cdot |\tau_i|^2 \ll \frac{1}{\varepsilon^2} \int_0^\varepsilon \big(o(1) \cdot \sqrt{v}\big)^2\, \mathrm{d}v = \frac{1}{\varepsilon^2} \cdot o(1) \cdot \frac{\varepsilon^2}{2} \longrightarrow 0$$

Thus, taking the ε_i sufficiently small, the smooth truncations $\Psi_{\varphi_i} - \Psi_{g_{\varepsilon_i} \cdot \tau_i}$ of the Ψ_{φ_i} are in $\mathfrak{E}_c^\infty \cap \mathfrak{E}_\Theta^0$, and still converge to f in $\mathfrak{E}^1$. *Q.E.D.*

Corollary 21. *An orthogonal basis for $\mathfrak{E}_\Theta^0$ consisting of $\widetilde{S}_\Theta$-eigenfunctions is an orthogonal basis for $\mathfrak{E}^1 \cap \mathfrak{E}_\Theta^0$, as well. In particular, for eigenfunction f with eigenvalue λ, we have $\langle f, f \rangle_{\mathfrak{E}^1} = \lambda \cdot \langle f, f \rangle$.*

Proof. Since $\widetilde{S}_\Theta^{-1}(\mathfrak{E}^0_\Theta) \supset \mathfrak{E}^\infty_c \cap \mathfrak{E}^0_\Theta$, by the theorem $\widetilde{S}_\Theta^{-1}(\mathfrak{E}^0_\Theta)$ is dense in $\mathfrak{E}^1 \cap \mathfrak{E}^0_\Theta$. Since finite linear combinations of the $\widetilde{S}_\Theta$-eigenfunctions are dense in $\mathfrak{E}^0_\Theta$ and $\widetilde{S}_\Theta^{-1}$ is continuous, their images in $\mathfrak{E}^1 \cap \mathfrak{E}^0_\Theta$ are dense there. From Proposition 1, for two $\widetilde{S}_\Theta$-eigenfunctions u, v with (real) eigenvalues λ, μ,

$$\langle u, v\rangle_{\mathfrak{E}^1} = \langle S^\#_\Theta u, v\rangle_{\mathfrak{E}^{-1}\times V} = \langle \widetilde{S}_\Theta u, v\rangle_{\mathfrak{E}^{-1}\times V} = \langle \widetilde{S}_\Theta u, v\rangle = \lambda\langle u, v\rangle$$

Symmetrically, $\langle u, v\rangle_{\mathfrak{E}^1} = \mu\langle u, v\rangle$. Thus, orthogonality of eigenfunctions in $\mathfrak{E}^0_\Theta \subset V$ implies orthogonality in $\mathfrak{E}^1 \cap \mathfrak{E}^0_\Theta \subset \mathfrak{E}^1$, and for $u = v$ we have $\langle u, u\rangle_{\mathfrak{E}^1} = \lambda \cdot \langle u, u\rangle$. *Q.E.D.*

Corollary 22. *Let $\{u_k : k = 1, 2, \ldots\}$ be the eigenfunctions for $\widetilde{S}_\Theta$, with eigenvalues λ_k. For $f \in \mathfrak{E}^1 \cap \mathfrak{E}^0_\Theta$,*

$$f = \sum_{k\geq 1} \langle f, u_k\rangle_V \cdot \frac{u_k}{\langle u_k, u_k\rangle_V} = \sum_{k\geq 1} \langle f, u_k\rangle_{\mathfrak{E}^1} \cdot \frac{u_k}{\langle u_k, u_k\rangle_{\mathfrak{E}^1}}$$

and these expansions converge to f not only in $\mathfrak{E}^0_\Theta$, but also in the finer topology of $\mathfrak{E}^1 \cap \mathfrak{E}^0_\Theta$.

Proof. Again, by the theorem, since $f \in \mathfrak{E}^1 \cap \mathfrak{E}^0_\Theta$ is in the closure of $\mathfrak{E}^\infty_c \cap \mathfrak{E}^0_\Theta$, and $\{u_k\}$ is an orthogonal basis for $\mathfrak{E}^1 \cap \mathfrak{E}^0_\Theta$, such f has an expansion

$$f = \sum_{k\geq 1} \langle f, u_k\rangle_{\mathfrak{E}^1} \cdot \frac{u_k}{\langle u_k, u_k\rangle_{\mathfrak{E}^1}}$$

convergent in $\mathfrak{E}^1 \cap \mathfrak{E}^0_\Theta$. As in the previous proof, from Proposition 1,

$$\langle f, u_k\rangle_{\mathfrak{E}^1} = \langle f, S^\#_\Theta u_k\rangle_{V\times\mathfrak{E}^{-1}} = \langle f, \widetilde{S}_\Theta u_k\rangle_{V\times\mathfrak{E}^{-1}} = \langle f, \widetilde{S}_\Theta u_k\rangle = \lambda_k\langle f, u_k\rangle$$

Thus, for every u_k,

$$\langle f, u_k\rangle_{\mathfrak{E}^1} \cdot \frac{u_k}{\langle u_k, u_k\rangle_{\mathfrak{E}^1}} = \lambda_k\langle f, u_k\rangle \cdot \frac{u_k}{\lambda_k\langle u_k, u_k\rangle} = \langle f, u_k\rangle \cdot \frac{u_k}{\langle u_k, u_k\rangle}$$

giving the termwise equality of the expansions. *Q.E.D.*

4.4 Extensions of restrictions by Eisenstein-Heegner and constant-term constraints

Fix a finite real-linear combination θ of Eisenstein-Heegner distributions $\theta_d \in \mathfrak{E}^{-\frac{1}{2}-\varepsilon}$. Fix $a > 1$. Let $S_{\theta,a}$ be $-\Delta$ restricted to domain

$$\mathfrak{E}_c^\infty \cap \ker\theta \cap \ker\eta_a.$$

Note that $S_{\theta,a} \geqslant \frac{1}{4}$ on the continuous spectrum, and its domain excludes constants. Symmetry of $S_{\theta,a}$ is inherited from S. The *pseudo-Laplacian* $\widetilde{S}_{\theta,a}$ is the Friedrichs extension of $S_{\theta,a}$ on $\mathfrak{E}^1 \cap \ker\theta \cap \ker\eta_a$, with $\widetilde{S}_{\theta,a} \geqslant 0$.

Theorem 23. *Given a finite real-linear combination θ of Eisenstein-Heegner distributions θ_d, for all a with $\Im z \neq a$ for all the Heegner points z involved, the Friedrichs extension $\widetilde{S}_{\theta,a}$ ignores θ and η_a, in the sense that for u in the domain of $\widetilde{S}_{\theta,a}$ the eigenvector condition $(\widetilde{S}_{\theta,a} - \lambda_w)\, u = 0$ is equivalent to the satisfaction of the equation*

$$(S^{\#}_{\theta,a} - \lambda_w)\, u = A \cdot \theta + B \cdot \eta_a \quad \text{(for some } A, B \in \mathbb{C}).$$

Proof. The point is to show that $(\mathbb{C}\cdot\theta \cap \eta_a) \cap (j^* \cap \Lambda)\mathfrak{E}^0 = \{0\}$. Then Theorem 4 applies.

There is an unique *highest* Heegner point z_0 appearing in θ. In fact, if $4|d$ the highest Heegner point in θ_d is $i\sqrt{|d|}/2$, while otherwise it is $(1 + i\sqrt{|d|})/2$. Then it suffices to take pseudo-Eisenstein series $f_n = \Psi_{\varphi_n}$ with $\varphi_n(\Im z_0) = 1$ and $\varphi_n(\Im z') = 1$ for the other Heegner points z' appearing in θ. Thus, $A \cdot \theta + B \cdot \eta_a$ is not in $\mathfrak{E}^0$ for $A \neq 0$. Thus, it suffices to show that η_a is not in $\mathfrak{E}^0$, which is even simpler. *Q.E.D.*

5 Eigenfunctions of pseudo-Laplacians

5.1 Determination of eigenfunctions

We continue to keep fixed a finite real-linear combination θ of Eisenstein-Heegner distributions θ_d. From the preceding Theorem 23, a solution $u \in \mathfrak{E}^1$ of an equation

$$(S^{\#} - \lambda_w)u = A \cdot \theta + B \cdot \eta_a$$

is a $\widetilde{S}_{\theta,a}$*-eigenfunction* precisely when $u \in \ker\theta \cap \ker\eta_a$. As above, let $v_{w,a}$ be the unique solution in $\mathfrak{E}^{\frac{3}{2}-\varepsilon}$ to $(-\Delta - \lambda_w) = \eta_a$ and similarly let $u_{\theta,w}$ be the unique solution in $\mathfrak{E}^{1+\varepsilon}$ to $(-\Delta - \lambda_w)u = \theta$. The condition for existence of a non-zero solution (A, B) to the homogeneous system

$$\begin{cases} \theta(Au_{\theta,w} + Bv_{a,w}) & = 0 \\ \eta_a(Au_{\theta,w} + Bv_{a,w}) & = 0 \end{cases} \tag{5.1}$$

is the vanishing of the determinant:

$$\det\begin{pmatrix} \theta(u_{\theta,w}) & \theta(v_{w,a}) \\ \\ \eta_a(u_{\theta,w}) & \eta_a(v_{w,a}) \end{pmatrix} = 0 \tag{5.2}$$

We compute the components.

5.2 Computing $\eta_a(v_{w,a})$ for $a > 1$ and $\Re(w) > \frac{1}{2}$

By the spectral expansion (3.14), this is

$$\begin{aligned} \eta_a(v_{w,a}) &= \frac{1}{\langle 1,1\rangle} + \frac{1}{4\pi i}\int_{(\frac{1}{2})}(a^{1-s} + c_{1-s}a^s)(a^s + c_s a^{1-s})\frac{\mathrm{d}s}{\lambda_s - \lambda_w} \\ &= \frac{1}{\langle 1,1\rangle} + \frac{1}{4\pi i}\int_{(\frac{1}{2})}(a + c_{1-s}a^{2s} + c_s a^{2-2s} + a)\frac{\mathrm{d}s}{\lambda_s - \lambda_w} \quad (\text{use } c_s c_{1-s} = 1) \\ &= \frac{1}{\langle 1,1\rangle} + \frac{1}{2\pi i}\int_{(\frac{1}{2})}(a + c_s a^{2-2s})\frac{\mathrm{d}s}{\lambda_s - \lambda_w} \quad (s \to 1-s \text{ in one term}) \end{aligned} \tag{5.3}$$

The behavior of c_s as a function of s in the half-plane $\sigma \geqslant \frac{1}{2}$ is easily determined from the second formula in (3.5). We find that c_s has a simple pole at $s = 1$ with residue $\dfrac{3}{\pi} = \dfrac{1}{\langle 1,1\rangle}$ and is of order $\sqrt{\sigma}(|t|+1)^{\frac{1}{2}+\varepsilon}$ for any fixed $\varepsilon > 0$, uniformly in σ and t. For $a > 1$, $|a^{2-2s}| = a^{2-2\sigma}$ goes exponentially to 0 (uniformly in t) as $\sigma \to +\infty$, so we can compute the integral in (5.3) by moving the line of integration to $\sigma = +\infty$ and see that it tends to 0 as $\sigma \to +\infty$. In doing so, we encounter residues at $s = w$ and $s = 1$. The residue at $s = 1$ cancels the constant term $1/\langle 1,1\rangle$. Noting that $\lambda_s - \lambda_w = -(s-w)(s-(1-w))$, and noting the negative orientation around $s = w$ of the path integral, the final result is

Theorem 24. *For* $a > 1$ *and* $\Re(w) > \frac{1}{2}$,

$$\eta_a(v_{w,a}) = \frac{a^{1-w}(a^w + c_w a^{1-w})}{2w-1}$$

5.3 Computing $\theta_d(v_{w,a})$ for $a \gg_\theta 1$ and $\Re(w) > \frac{1}{2}$

By linearity, it suffices to compute $\delta_z^{\mathrm{nc}}(v_{w,a})$ when z is a Heegner point. Note that $v_{w,a} \in \mathfrak{E}^{\frac{3}{2}-\varepsilon}$ for all $\varepsilon > 0$, so the integral for the pairing $\mathfrak{E}^{\frac{3}{2}-\varepsilon} \times \mathfrak{E}^{-1-\varepsilon}$ is absolutely convergent if ε is sufficiently small.

Using $c_{1-s}E_s = E_{1-s}$ and the spectral expansion, we find

$$\begin{aligned}
\delta_z^{\mathrm{nc}}(v_{w,a}) &= \frac{1}{\langle 1,1\rangle} + \frac{1}{4\pi i}\int_{(\frac{1}{2})} \eta_a E_{1-s}(z)\cdot E_s(z)\,\frac{\mathrm{d}s}{\lambda_s - \lambda_w} \\
&= \frac{1}{\langle 1,1\rangle} + \frac{1}{4\pi i}\int_{(\frac{1}{2})} (a^{1-s} + c_{1-s}a^s)\cdot E_s(z)\,\frac{\mathrm{d}s}{\lambda_s - \lambda_w} \\
&= \frac{1}{\langle 1,1\rangle} + \frac{1}{4\pi i}\int_{(\frac{1}{2})} (a^{1-s}E_s(z) + a^s E_{1-s}(z))\,\frac{\mathrm{d}s}{\lambda_s - \lambda_w} \qquad \text{(use } c_{1-s}E_s = E_{1-s}\text{)} \\
&= \frac{1}{\langle 1,1\rangle} + \frac{1}{2\pi i}\int_{(\frac{1}{2})} a^{1-s}E_s(z)\,\frac{\mathrm{d}s}{\lambda_s - \lambda_w} \qquad (5.4)
\end{aligned}$$

The computation of the integral requires some care, depending on the height of z relative to a. To this end, we proceed as before, moving the line of integration from $\sigma = \frac{1}{2}$ to $\sigma = C$ where $C > 1$ (the actual value of C is immaterial), thereby acquiring the contribution of residues at $s = w$ and also at $s = 1$ from the Eisenstein series. The residue of $E_s(z)$ at $s = 1$ is $\dfrac{3}{\pi} = \dfrac{1}{\langle 1,1\rangle}$, hence its contribution cancels the constant term and one obtains

$$\delta_z^{\mathrm{nc}}(v_{w,a}) = \frac{a^{1-w}E_w(z)}{2w-1} + \frac{1}{2\pi i}\int_{(C)} a^{1-s}E_s(z)\,\frac{\mathrm{d}s}{\lambda_s - \lambda_w} \qquad (5.5)$$

The series for $E_s(z)$ with $z = x+iy$ is absolutely convergent for $\Re(s) = c > 1$ and for $y \to \infty$ it is asymptotic to y^s. It is obvious that for $\sigma > 1$ we have

$$|E_s(z)| \leqslant \frac{1}{2}\sum_{m,n}{}' \frac{y^\sigma}{|nz+n|^{2\sigma}}$$

where the dash means that the sum is extended to all pairs (m,n) of coprime integers. It follows that if y is bounded away from zero (in our case $y \geqslant \sqrt{3}/2$) then $|E_s(z)| \ll (\max(1, y^\sigma)$ uniformly for $\sigma \geqslant c > 1$, with the constant involved in the inequality depending only on c.

Therefore, if $y/a < 1$ one may move the line of integration all the way to $+\infty$, showing that the integral in question vanishes. We have proved that in this case

$$\delta_z^{\mathrm{nc}}(v_{w,a}) = \frac{a^{1-w} E_w(z)}{2w-1} \qquad (\Im(z) < a) \tag{5.6}$$

If instead $y/a > 1$ the analysis is more complicated. In this case, we split the sum for $E_s(z)$ into two components:

$$\begin{aligned} E_s(z) &= \frac{1}{2} \sum_{\substack{|mz+n|^2 > y/a \\ \mathrm{GCD}(m,n)=1}} \frac{y^s}{|mz+n|^{2s}} + \frac{1}{2} \sum_{\substack{|mz+n|^2 \leqslant y/a \\ \mathrm{GCD}(m,n)=1}} \frac{y^s}{|mz+n|^{2s}} \\ &= \Sigma_1 + \Sigma_2. \end{aligned} \tag{5.7}$$

The evaluation of the integral $\displaystyle\frac{1}{2\pi i}\int_{(C)} a^{1-s}\,\Sigma_1 \frac{\mathrm{d}s}{\lambda_s - \lambda_w}$ can be done as before by letting $C \to +\infty$, obtaining

$$\frac{1}{2\pi i}\int_{(C)} a^{1-s}\,\Sigma_1 \frac{\mathrm{d}s}{\lambda_s - \lambda_w} = 0. \tag{5.8}$$

To deal with the integral involved in Σ_2 we note that the sum involved is a finite sum. For $|cz+d|^2 \neq y/a$ we can integrate term-by-term and move the line of integration backwards all the way to $\to -\infty$, with the limit of the integral being 0. In doing this we encounter two residues at $s = w$ and $s = 1-w$, and conclude that

$$\begin{aligned} \frac{1}{2\pi i}\int_{(C)} a^{1-s}\,\Sigma_2 \frac{\mathrm{d}s}{\lambda_s - \lambda_w} &= -\frac{1}{2}\frac{a^{1-w}}{2w-1} \sum_{\substack{|mz+n|^2 \leqslant y/a \\ \mathrm{GCD}(m,n)=1}} \frac{y^w}{|mz+n|^{2w}} \\ &\quad + \frac{1}{2}\frac{a^{w}}{2w-1} \sum_{\substack{|mz+n|^2 \leqslant y/a \\ \mathrm{GCD}(m,n)=1}} \frac{y^{1-w}}{|mz+n|^{2-2w}}. \end{aligned} \tag{5.9}$$

We have proved: If $\Im(z) < a$ then

$$\delta_z^{\mathrm{nc}}(v_{w,a}) = \frac{a^{1-w}E_w(z)}{2w-1}. \tag{5.10}$$

while if $y = \Im(z) \geqslant a$ then

$$\begin{aligned}\delta_z^{\mathrm{nc}}(v_{w,a}) = \frac{a^{1-w}E_w(z)}{2w-1} - \frac{1}{2}\frac{a^{1-w}}{2w-1} \sum_{\substack{|mz+n|^2<y/a \\ \mathrm{GCD}(m,n)=1}} \frac{y^w}{|mz+n|^{2w}} \\ +\frac{1}{2}\frac{a^w}{2w-1} \sum_{\substack{|mz+n|^2<y/a \\ \mathrm{GCD}(m,n)=1}} \frac{y^{1-w}}{|mz+n|^{2-2w}}.\end{aligned} \tag{5.11}$$

The condition $|mz+n|^2 < y/a$ is rather restrictive on the pair (m,n). In fact, if $m \neq 0$ then $|mz+n|^2 \geqslant y^2$ and the condition $(my)^2 < y/a$ implies $y < 1/(m^2a)$, which implies $m = \pm 1$ or $m = 0$. If $m = \pm 1$ then $|mz+n|^2 = (x \mp n)^2 + y^2 < y/a$, hence $\frac{1}{4} + y^2 < y$ because $|x \mp n| \geqslant \frac{1}{2}$ and $a > 1$ by hypothesis, but this is impossible. We conclude that $m = 0$, $n = \pm 1$ and we get the much simplified formula

$$\delta_z^{\mathrm{nc}}(v_{w,a}) = \frac{a^{1-w}E_w(z)}{2w-1} - \frac{a^{1-w}y^w - a^w y^{1-w}}{2w-1} \tag{5.12}$$

in the remaining range $1 < a < y$. By linearity, this extends to the computation of $\theta_d(v_{w,a})$ for Eisenstein-Heegner distributions θ_d, and to real-linear combinations of such. We summarize these computations as a theorem:

Theorem 25. *Let $a > 1$, and let $d < 0$ be a fundamental discriminant. Then*

$$\theta_d(v_{w,a}) = \frac{1}{2w-1}\left\{a^{1-w}\theta_d E_w - R_w(d,a)\right\} \tag{5.13}$$

where

$$R_w(d,a) = \sum_{\substack{x+iy\in H_d \\ y>a}} (a^{1-w}y^w - a^w y^{1-w}) \tag{5.14}$$

5.4 Computing $\eta_a(u_{\theta,w})$ for $a \gg_\theta 1$ and $\Re(w) > \frac{1}{2}$

Let θ be a finite real-linear combination of Eisenstein-Heegner distributions θ_d.

Theorem 26.

$$\eta_a(u_{\theta,w}) = \theta(v_{w,a}). \tag{5.15}$$

Proof. One computes

$$\begin{aligned}
\eta_a(u_{\theta,w}) &= \frac{\eta_a(1)\theta(1)}{\langle 1,1\rangle} + \frac{1}{4\pi i}\int_{(\frac{1}{2})} \theta E_{1-s}\cdot \eta_a E_s \,\frac{\mathrm{d}s}{\lambda_s-\lambda_w} \\
&= \frac{\theta(1)}{\langle 1,1\rangle} + \frac{1}{4\pi i}\int_{(\frac{1}{2})} \theta E_{1-s}\cdot (a^s + c_s a^{1-s})\,\frac{\mathrm{d}s}{\lambda_s-\lambda_w} \\
&= \frac{\theta(1)}{\langle 1,1\rangle} + \frac{1}{4\pi i}\int_{(\frac{1}{2})} (\theta E_{1-s}\cdot a^s + \theta E_s\cdot a^{1-s})\,\frac{\mathrm{d}s}{\lambda_s-\lambda_w} \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{(because } c_s E_{1-s} = E_s\text{)} \\
&= \frac{\theta(1)}{\langle 1,1\rangle} + \frac{1}{2\pi i}\int_{(\frac{1}{2})} \theta E_{1-s}\cdot a^s\,\frac{\mathrm{d}s}{\lambda_s-\lambda_w}. \\
&\qquad\qquad\qquad\qquad \text{(by changing } s\to 1-s \text{ in one term)}
\end{aligned}$$

The theorem follows by linearity and equation (5.4). *Q.E.D.*

5.5 Computing $\theta(u_{\theta,w})$ for $a > 1$ and $\Re(w) > \frac{1}{2}$

The outcome here does not admit much simplification, in contrast to the other cases. That is, from the spectral expansions of subsections 3.6 and 3.7, via the $\mathfrak{E}^1 \times \mathfrak{E}^{-1}$ pairing of 3.4, we obtain

Theorem 27. *For θ a finite real-linear combination of Eisenstein-Heegner distributions θ_d,*

$$\theta(u_{\theta,w}) = \frac{|\theta(1)|^2}{\langle 1,1\rangle\cdot(\lambda_1-\lambda_w)} + \frac{1}{4\pi i}\int_{(\frac{1}{2})} |\theta E_s|^2\,\frac{\mathrm{d}s}{\lambda_s-\lambda_w}. \tag{5.16}$$

5.6 Rewriting the determinant condition

First, by Theorem 5.15 the determinant-vanishing condition (5.2) becomes

$$\eta(v_{w,a})\theta(u_{\theta,w}) - \eta_a(u_{\theta,w})^2 = 0$$

In view of Theorems 24, 25, 27,

Corollary 28. *Let $\theta = \sum_d \nu_d \theta_d$ be a finite real-linear combination of Eisenstein-Heegner distributions θ_d with $d < -4$. For all $a > 1$, all w with $\Re(w) > \frac{1}{2}$ and off $(\frac{1}{2}, 1]$,*

$$a^{1-w}(a^w + c_w a^{1-w}) \times \Big(\frac{|\theta(1)|^2}{\langle 1,1\rangle \cdot \lambda_1 - \lambda_w} + \frac{1}{4\pi i} \int_{(\frac{1}{2})} |\theta E_s|^2 \frac{\mathrm{d}s}{\lambda_s - \lambda_w} \Big)$$
$$\neq \frac{1}{2w-1} \left(a^{1-w} \theta E_w - R_w(\theta, a) \right)^2 \tag{5.17}$$

where

$$\theta E_s = \sum_d \nu_d \left(\frac{\sqrt{|d|}}{2} \right)^s \frac{\zeta(s)}{\zeta(2s)} L(s, \chi_d) \tag{5.18}$$

and where

$$R_w(\theta, a) = \sum_d \nu_d \sum_{\substack{x+iy \in H_d \\ y > a}} (a^{1-w} y^w - a^w y^{1-w}). \tag{5.14}$$

Proof. In $\Re(w) > \frac{1}{2}$, $u_{\theta,w}$ and $v_{a,w}$ are in $\mathfrak{E}^1$. The vanishing condition $\eta(v_{w,a})\theta(u_{\theta,w}) - \eta_a(u_{\theta,w})^2 = 0$ is thus necessary and sufficient for some non-zero linear combination of $u_{\theta,w}$ and $v_{a,w}$ to be an eigenfunction for the self-adjoint semibounded operator $\widetilde{S}_{\theta,a}$, with eigenvalue $\lambda_w = w(1-w)$. Such an eigenvalue must be real and satisfy $0 \leqslant \lambda_w$. *Q.E.D.*

5.7 Meromorphic continuation and location of zeros

Here we give a direct, relatively elementary argument for meromorphic continuation of the two-by-two determinant above. Theorem 43 in section 6.5 gives several stronger results by less elementary means, useful in the discussion of unconditional results on zero spacing in Section 8. For brevity, write

$$F(a, w) := (a^w + c_w a^{1-w}) \Big(\frac{|\theta(1)|^2}{\langle 1,1\rangle \cdot (\lambda_1 - \lambda_w)} + \frac{1}{4\pi i} \int_{(\frac{1}{2})} |\theta E_s|^2 \frac{\mathrm{d}s}{\lambda_s - \lambda_w} \Big)$$
$$- \frac{a^{1-w}\, \theta(E_w)^2}{2w-1} \tag{5.19}$$

for the determinant above, where for the time being $\Re(w) > \frac{1}{2}$ and a is real with $a > 1$.

The analytic continuation of $F(a,w)$ beyond the line $\Re(w) = \frac{1}{2}$ is easily accomplished. Except for a possible simple pole at $w = 1$, the function $s \to \theta E_s$ is holomorphic for $\Re(s) > \frac{1}{2} - C/\log(2 + |s|)$ for some absolute positive constant C. Hence, when $\frac{1}{2} < \Re(w) < \frac{1}{2} + C/\log(2 + |w|)$ we can evaluate the integral in (5.19) as follows :

$$\begin{aligned}\frac{1}{4\pi i}\int_{(\frac{1}{2})} |\theta E_s|^2 \frac{\mathrm{d}s}{\lambda_s - \lambda_w}&\\ &= \frac{1}{4\pi i}\int_{(\frac{1}{2})} \Big(\theta E_s \cdot \theta E_{1-s} - \theta E_w \cdot \theta E_{1-w}\Big) \frac{\mathrm{d}s}{\lambda_s - \lambda_w}\\ &\quad + \theta E_w \cdot \theta E_{1-w} \frac{1}{4\pi i}\int_{(\frac{1}{2})} \frac{\mathrm{d}s}{\lambda_s - \lambda_w}\,. \end{aligned} \tag{5.20}$$

The left-hand side of this equation is a holomorphic function for $\Re(w) > \frac{1}{2}$. For $\Re(w) > \frac{1}{2}$ the last integral is evaluated by the calculus of residues, moving the line of integration to $\Re(s) \to +\infty$:

$$\frac{1}{4\pi i}\int_{(\frac{1}{2})} \frac{\mathrm{d}s}{\lambda_s - \lambda_w} = \frac{1}{2(2w-1)}. \tag{5.21}$$

Therefore, for $\frac{1}{2} < \Re(w) < \frac{1}{2} + C/(2 + |w|)$ we have

$$\begin{aligned} F(w,a) &= (a^w + c_w a^{1-w}) \frac{|\theta(1)|^2}{\langle 1,1\rangle \cdot (\lambda_1 - \lambda_w)}\\ &\quad + (a^w + c_w a^{1-w}) \frac{1}{4\pi i}\int_{(\frac{1}{2})} \Big(\theta E_s \cdot \theta E_{1-s} - \theta E_w \cdot \theta E_{1-w}\Big) \frac{\mathrm{d}s}{\lambda_s - \lambda_w}\\ &\quad + (a^w + c_w a^{1-w}) \frac{\theta E_w \cdot \theta E_{1-w}}{2(2w-1)} - \frac{a^{1-w}\theta(E_w)^2}{2w-1} \end{aligned} \tag{5.22}$$

Using the functional equation $c_w \theta E_{1-w} = \theta E_w$ simplifies this into

$$\begin{aligned} F(w,a) &= (a^w + c_w a^{1-w}) \frac{|\theta(1)|^2}{\langle 1,1\rangle \cdot (\lambda_1 - \lambda_w)}\\ &\quad + (a^w + c_w a^{1-w}) \frac{1}{4\pi i}\int_{(\frac{1}{2})} \Big(\theta E_s \cdot \theta E_{1-s} - \theta E_w \cdot \theta E_{1-w}\Big) \frac{\mathrm{d}s}{\lambda_s - \lambda_w}\\ &\quad + (a^w - c_w a^{1-w}) \frac{\theta E_w \cdot \theta E_{1-w}}{2(2w-1)} \end{aligned} \tag{5.23}$$

This formula, so far proved for $\frac{1}{2} < \Re(w) < \frac{1}{2} + C/\log(2+|w|)$, extends to an open neighborhood of the line $\Re(w) = \frac{1}{2}$ in the complex plane $w \in \mathbb{C}$, because now the integral is well defined there as a continuous function of w. Indeed, the numerator of the integrand vanishes as $w \to s$ with $\Re(s) = \frac{1}{2}$ at least to the first order when $s \neq \frac{1}{2}$ and at at least to the second order when $s = \frac{1}{2}$, while the growth of $\theta E_s \cdot \theta E_{1-s}$ is of order not more than $|s|^{1-\delta}$ there for some fixed $\delta > 0$ if the neighborhood is sufficiently small, hence the integral is absolutely convergent.

Theorem 29. *With $a > 1$, let*

$$G(w,a) := \frac{F(w,a)}{a^w + c_w a^{1-w}} \tag{5.24}$$

Then $G(w,a)$ is a meromorphic function in the whole complex w-plane and satisfies the functional equation

$$G(w,a) = G(1-w,a). \tag{5.25}$$

Proof. A simple computation using the expansion (5.7) shows that the stated functional equation holds in an open neighborhood of the critical line. Due to the importance of this symmetry, we carry out this computation in detail.

$$\begin{aligned} G(w,a) = {} & \frac{|\theta(1)|^2}{\langle 1,1\rangle \cdot (\lambda_1 - \lambda_w)} \\ & + \frac{1}{4\pi i} \int_{(\frac{1}{2})} \Big(\theta E_s \cdot \theta E_{1-s} - \theta E_w \cdot \theta E_{1-w}\Big) \frac{\mathrm{d}s}{\lambda_s - \lambda_w} \\ & \qquad + \frac{a^w - c_w a^{1-w}}{a^w + c_w a^{1-w}} \frac{\theta E_w \cdot \theta E_{1-w}}{2(2w-1)} \end{aligned} \tag{5.26}$$

The first two summands are indeed invariant under $w \to 1-w$, as is $\theta E_w \cdot \theta E_w$ in the third summand. Finally, under $w \to 1-w$, using $c_w \cdot c_{1-w} = 1$, the part

$$\frac{a^w - c_w a^{1-w}}{a^w + c_w a^{1-w}} \frac{1}{2w-1}$$

of the third summand becomes

$$\frac{a^{1-w} - c_{1-w} a^w}{a^{1-w} + c_{1-w} a^w} \frac{-1}{2w-1} = \frac{c_w a^{1-w} - a^w}{c_w a^{1-w} + a^w} \frac{-1}{2w-1} = \frac{a_w - c_w a^{1-w}}{a^w + c_w a^{1-w}} \frac{1}{2w-1}$$

giving the claimed invariance. The conclusion of the theorem follows by analytic continuation. *Q.E.D.*

Corollary 30. *The only zeros of the function $G(w,a)$ defined in equation 5.24 are on $\Re(w) = \frac{1}{2}$ and $[0,1]$.*

Proof. Corollary 28 shows that $G(w,a)$ cannot vanish in $\Re(w) > \frac{1}{2}$ except possibly on $(\frac{1}{2},1]$, because otherwise $\lambda_w = w(1-w)$ would be an eigenvalue for a non-negative self-adjoint operator. Then the symmetry of 5.25 shows non-vanishing in $\Re(w) < \frac{1}{2}$ except possibly on $[0,\frac{1}{2})$. *Q.E.D.*

5.8 An important remark

The preceding considerations also apply to a general real-linear combination $\theta = \sum_\nu b_\nu \cdot \delta^{\mathrm{nc}}_{z_\nu}$ of Eisenstein-Dirac distributions $\delta^{\mathrm{nc}}_{z_\nu}$. With the simplifying assumption $\sum b_\nu E_s(z_\nu) = 0$, we have

$$a^{1-w}(a^w + c_w a^{1-w}) \times \frac{1}{4\pi i}\int_{(\frac{1}{2})} \left|\sum b_\nu E_s(z_\nu)\right|^2 \frac{\mathrm{d}s}{\lambda_s - \lambda_w}$$
$$= \frac{1}{1-2w}\left\{\sum b_\nu \left[a^{1-w}E_w(z_\nu) - 2\sqrt{a\Im(z_\nu)}\sinh\left(\frac{1}{2}(1-w)\log^+\frac{\Im(z_\nu)}{a}\right)\right]\right\}^2 \tag{5.27}$$

where $\log^+ x = \max(\log x, 0)$.

If $a > \max\Im(\zeta_\nu)$ then for almost all a the vanishing of $\sum b_\nu E_w(z_\nu)$ implies the vanishing of the left-hand term, which means either $a^{1-w}(a^w + c_w a^{1-w}) = 0$, which vanishes only when $\Re(s) = \frac{1}{2}$, or

$$\frac{1}{4\pi i}\int_{(\frac{1}{2})} \left|\sum b_\nu E_s(z_\nu)\right|^2 \frac{\mathrm{d}s}{\lambda_s - \lambda_w} = 0.$$

Thus if zeros on the critical line of a function $\sum_\nu b_\nu E_s(z_\nu)$ with $\sum b_\nu = 0$ had a spectral interpretation for some $a = a_0 > \max\Im(z_\nu)$ this would be so for all $a > a_0$, implying that the corresponding eigenvalues $w(1-w)$ would be independent of a. For $a \to \infty$ this is analogous to the condition formulated by Colin de Verdière in the special case $\{\nu\} = \{1\}$, $b_1 = 1$, $z_1 = \frac{1}{2} + i\frac{\sqrt{3}}{2}$, and, tentatively, suggested by him as a spectral intepretation of the zeros of $\zeta_{Q(\sqrt{-3})}(s)$ on the critical line.

5.9 Computing θE_s in a special case

The fundamental discriminants d are the odd squarefree numbers $d = m$ with $m \equiv 1 \pmod 4$ or numbers of the type $d = 4m$ with m squarefree and $m \equiv 2,3 \pmod 4$. To m, one associates the real primitive character χ_m given by

$$\chi_m(n) = \begin{cases} \left(\dfrac{m}{n}\right) & m \equiv 1 \,(\mathrm{mod} 4) \\ \\ \left(\dfrac{4m}{n}\right) & m \equiv 2,3 \,(\mathrm{mod} 4) \end{cases} \tag{5.28}$$

where on the right-hand side we have the Kronecker symbol. Then $\zeta(s)L(s,\chi_m)$ is the zeta function of the quadratic field $\mathbb{Q}(\sqrt{d})$.

A precise asymptotic formula for a certain simple linear combination of quadratic L-functions has been obtained in the paper [12] of Goldfeld and Hoffstein. We recall *verbatim* their Theorem (1), where their reference (0.6) is our equation (5.28). (Compare also [28].)

Theorem (1) *Let $\varepsilon > 0$ be fixed. Let χ_m be defined as in (0.6). Then there exist analytic functions $c(\rho)$ and $c^*_\pm(\rho)$ with Laurent expansion $c(\rho) = c_{\frac12}/(\rho - \frac12) + c'_{\frac12} + O(\rho - \frac12)$, $c^*_\pm(\rho) = -c_{\frac12}$ such that*

$$\sum_{\substack{1<-m<x \\ m \ \square-free}} L(\rho,\chi_m)$$

$$= \begin{cases} c(\rho)x + O(x^{\frac12+\varepsilon}) & if \ \ \mathrm{Re}(\rho) \geq 1 \\ c(\rho)x + c^*_\pm(\rho)x^{\frac32-\rho} + O(x^{\theta+\varepsilon}) & if \ \ \mathrm{Re}(\rho) \neq \frac12, \ \frac12 \leq \mathrm{Re}(\rho) < 1 \\ c_{\frac12}\, x \log x + (c'_{\frac12} + c^{*\prime}_{\pm\frac12} - c_{\frac12})x + O(x^{\frac{19}{32}+\varepsilon}) & if \ \ \rho = \frac12. \end{cases}$$

Here

$$c(\rho) = \frac34(1-2^{-2\rho})\zeta(2\rho)\prod_{p\neq 2}(1 - p^{-2} - p^{-2\rho-1} + p^{-2\rho-2}),$$

$$c_{\frac12} = \frac{3}{16}\prod_{p\neq 2}(1 - 2p^{-2} + p^{-3})$$

and

$$\theta = \begin{cases} \dfrac{1}{2} & if \;\; \mathrm{Re}(\rho) > \dfrac{-5+\sqrt{193}}{12} \\ \\ \dfrac{19+3\mathrm{Re}(\rho)-6\mathrm{Re}(\rho)^2}{24+16\mathrm{Re}(\rho)} & if \;\; \dfrac{1}{2} \leqq \mathrm{Re}(\rho) \leqq \dfrac{-5+\sqrt{193}}{12} \end{cases}$$

and all O-constants depend at most on ρ, ε.

The authors do not give the dependence on ρ in the proportionality factor involved in the symbol $O(...)$, but there must be a function $\omega(x)$ slowly increasing to ∞ such that the estimates remain uniform in ρ as long as $|\Im(\rho)| < \omega(x)$.

What is of interest to us is not the sum $\sum L(\rho, \chi_m)$ (with $m < 0$) but rather the sum divided by $\zeta(2\rho)$. This yields the following result.

Let

$$A(s) = \frac{3}{4}(1-2^{-2s})\prod_{p\neq 2}(1-p^{-2}-p^{-2s-1}+p^{-2s-2}) \tag{5.29}$$

Theorem 31. *There is a function* $\omega(x)$*, slowly increasing to* ∞ *as* $x \to \infty$*, such that*

$$\sum_{\substack{1<-m<x \\ m\ \Box-free}} \zeta(2s)^{-1} L(s,\chi_m) = A(s)x + B(s)x^{\frac{3}{2}-s} + O(x^{\frac{19}{32}+\varepsilon}) \tag{5.30}$$

holds, uniformly for $\frac{1}{2} \leqslant \Re(s) \leqslant 1$ *and* $|\Im(s)| < \omega(x)$.

The function $B(s)$ is more complicated to describe but it is holomorphic for $\frac{1}{2} \leqslant \Re(s)$. (Again, compare [28].)

6 Meromorphic continuations of spectral integrals

Here we prove meromorphic continuation results stronger than the scalar meromorphic continuation in Theorem 29. The immediate goal is to prove that, for suitable $\theta \in \mathfrak{E}^{-1+\varepsilon}$, solutions $u_{\theta,w}$ of equations $(-\Delta - \lambda_w)u = \theta$ expressed by spectral expansions

$$u_{\theta,w} = \frac{\langle\theta,1\rangle \cdot 1}{(\lambda_1-\lambda_w)\cdot\langle 1,1\rangle} + \frac{1}{4\pi i}\int_{(\frac{1}{2})} \frac{\mathcal{E}\theta(s)\cdot E_s}{\lambda_s-\lambda_w}\,\mathrm{d}s$$

convergent in $\mathfrak{E}^{1+\varepsilon}$, at first valid only in $\Re(w) > \frac{1}{2}$, meromorphically extend to the whole complex plane, as function-valued functions. However, the meromorphic continuations do not lie in $\mathfrak{E}^{1+\varepsilon}$, but only in a larger space M, large enough to include Eisenstein series. On the critical line, we find that $u_{\theta,w}$ stays in $\mathfrak{E}^{1+\varepsilon}$ only for $\mathcal{E}\theta(w) = 0$. By Corollary 11, the only *possible* discrete spectrum $\lambda_w > \frac{1}{4}$ of $\widetilde{S}_\theta$ occurs among zeros of $\mathcal{E}\theta$, this will show that eigenfunctions in the discrete spectrum, if any, are meromorphic continuations of these spectral synthesis integrals. However, this still does not prove that zeros w of $\mathcal{E}\theta$ on $\Re(w) = \frac{1}{2}$ gives eigenvalues $w(1-w)$ of $\widetilde{S}_\theta$, since $u_{\theta,w}$ is not in the domain of $\widetilde{S}_\theta$ unless, additionally, $\theta u_{\theta,w} = 0$. In fact, we discuss meromorphic continuations of images $\Phi u_{\theta,w}$ under continuous linear maps $\Phi : M \to N$ for quasi-complete, locally convex topological vector spaces N.

6.1 Vector-valued integrals

We recall some standard results about vector-valued integrals, mostly without proofs. Original sources are [11] and [25], for which [26] offers a reasonable exposition. See also [10] chapter 14 for exposition and proofs more tightly aimed at applications such as those here. Let V be a topological vectorspace over $\mathbb{C}$, f a measurable V-valued function on a measure space X. A *Gelfand-Pettis integral* of f is a vector $I_f \in V$ so that $\lambda(I_f) = \int_X \lambda \circ f$ for all $\lambda \in V^*$. If it exists and is unique, this vector I_f is denoted $\int_X f$. In contrast to construction of integrals as limits, this characterization is a property no reasonable notion of integral would lack. Since this property is an irreducible minimum, this characterizes a *weak integral*.

Uniqueness of the integral is immediate when V^* separates points on V, as for locally convex V, by the Hahn-Banach theorem. Similarly, linearity of $f \to I_f$ follows when V^* separates points. Thus, the issue is existence.

The functions we integrate are relatively nice: compactly-supported and continuous, on measure spaces with finite, positive, regular Borel measures. In this situation, all the $\mathbb{C}$-valued integrals $\int_X \lambda \circ f$ exist for elementary reasons, being integrals of compactly-supported $\mathbb{C}$-valued continuous functions on a compact set with respect to a finite regular Borel measure.

A topological vector space is *quasi-complete* or *locally complete* if every *bounded* (in the general topological vector space sense) Cauchy net is convergent. It is known (for example, see [2]) that

Lemma 32. *In a quasi-complete, local convex topological vector space, the convex hull of a compact set has compact closure.*

The latter property ensures existence of certain Gelfand-Pettis integrals:

Theorem 33. *Let X be a locally compact Hausdorff topological space with a finite, positive, regular Borel measure. Let V be a locally convex topological vectorspace in which the closure of the convex hull of a compact set is compact. Then continuous, compactly-supported V-valued functions f on X have Gelfand-Pettis integrals. Further,*

$$\int_X f \;\in\; \mathit{meas}(X) \cdot \Big(\text{closure of convex hull of } f(X)\Big)$$

is the basic estimate substituting for estimating a Banach-space norm of an integral by the integral of the norm of the integrand.

The legitimacy of passing continuous operators inside such integrals is an easy corollary:

Corollary 34. *Let $T : V \to W$ be a continuous linear map of locally convex topological vectorspaces, where convex hulls of compact sets in V have compact closures. Let f be a continuous, compactly-supported V-valued function on a finite regular measure space X. Then the W-valued function $T \circ f$ has a Gelfand-Pettis integral, and $T\Big(\int_X f\Big) = \int_X T \circ f$.*

Proof. *(of corollary)* To verify that the left-hand side of the asserted equality fulfills the requirements of a Gelfand-Pettis integral of $T \circ f$, we must show that

$$\lambda\Big(\text{left-hand side}\Big) \;=\; \int_X \lambda \circ (T \circ f)$$

for all $\lambda \in W^*$. Starting with the left-hand side,

$$\lambda\Big(T\Big(\int_X f\Big)\Big) = (\lambda \circ T)\Big(\int_X f\Big) = \int_X (\lambda \circ T) \circ f = \int_X \lambda \circ (T \circ f)$$

proving that $T\big(\int_X f\big)$ is a weak integral of $T \circ f$. *Q.E.D.*

6.2 Holomorphic vector-valued functions

Now we recall basic facts about holomorphic vector-valued functions, mostly without proof, for which we refer to the original source [13], or expositions such as [26] or [10], chapter 14. Existence and properties of vector-valued integrals are ingredients in the proofs of the assertions below.

A function f on an open set $\Omega \subset \mathbb{C}$ taking values in a quasi-complete, locally convex topological vector space V is (strongly) *complex-differentiable* when

$$\lim_{z \to z_o} \frac{1}{z - z_o} \cdot \big(f(z) - f(z_o)\big)$$

exists (in V) for all $z_o \in \Omega$, where $z \to z_o$ specifically means for *complex* z approaching z_o. The function f is (strongly) *analytic* when it is locally expressible as a convergent power series with coefficients in V. The function f is *weakly holomorphic* when the $\mathbb{C}$-valued functions $\lambda \circ f$ are holomorphic for all λ in V^*.

Theorem 35. *For V a locally convex quasi-complete topological vector space, weakly holomorphic V-valued functions f are strongly holomorphic, in the following senses. First the usual Cauchy-theory integral formulas apply:*

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z} \, d\zeta$$

with γ a closed path around z having winding number $+1$. Second, the function $f(z)$ is infinitely differentiable, in fact strongly analytic, that is, expressible as a convergent power series $f(z) = \sum_{n \geq 0} c_n \, (z - z_o)^n$ with coefficients $c_n \in V$ given by Gelfand-Pettis integrals echoing Cauchy's formulas:

$$c_n = \frac{f^n(z_o)}{n!} = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z_o)^{n+1}} \, d\zeta$$

The appropriate vector-valued notion of *meromorphy* is completely parallel to the scalar-valued version: a V-valued function f defined on a punctured neighborhood N of z_o is *meromorphic* at z_o when there is a positive integer n such that $(z - z_o)^m \cdot f(z)$ has an extension to a V-valued *holomorphic* function on $N \cup \{z_o\}$.

Fix a non-empty open subset Ω of $\mathbb{C}$. Let V be quasi-complete, locally convex, with topology given by seminorms $\{\nu\}$. The space $\mathrm{Hol}(\Omega, V)$ of holomorphic V-valued functions on Ω has the natural topology given by seminorms $\mu_{\nu,K}(f) =$

$\sup_{z\in K} \nu(f(z))$ for compacts $K \subset \Omega$, seminorms ν on V. This is the analogue of the sups-on-compacts seminorms on scalar-valued holomorphic functions, and there is the analogous corollary of the vector-valued Cauchy formulas:

Corollary 36. $\mathrm{Hol}(\Omega, V)$ *is locally convex, quasi-complete.*

A V-valued function $f(z, w)$ on a non-empty open subset $\Omega \subset \mathbb{C}^2$ is *complex analytic* when it is locally expressible as a convergent power series in z and w, with coefficients in V. The two-variable version of the above discussion of power series with coefficients in V succeeds without incident.

Again by the vector-valued form of Cauchy's integral formulas:

Corollary 37. *Let V be quasi-complete, locally convex. Let $f(z, w)$ be complex-analytic V-valued in two variables, on a domain $\Omega_1 \times \Omega_2 \subset \mathbb{C}^2$. Then the function $w \longrightarrow (z \to f(z, w))$ is a holomorphic $\mathrm{Hol}(\Omega_1, V)$-valued function on Ω_2.*

There is a vector-valued version of Abel's theorem on convergent power series in one complex variable, proven in similar fashion:

Proposition 38. *Let c_n be a bounded sequence of vectors in a locally convex quasi-complete topological vector space V. Then on $|z| < 1$ the series $f(z) = \sum_n c_n z^n$ converges and gives a holomorphic V-valued function. That is, the function is infinitely-many-times complex-differentiable.*

6.3 Spaces M of moderate-growth functions

The space of moderate-growth continuous functions on $\Gamma\backslash\mathfrak{H}$ is

$$M = \{f \in C^0(\Gamma\backslash\mathfrak{H}) : \sup_{\Im(z)\geq\sqrt{3}/2} y^{-r} \cdot |f(x+iy)| < \infty, \text{ for some } r \in \mathbb{R}\}$$

The correct topology is as a strict inductive limit of Banach subspaces M_0^r each obtained as a completion of $C_c^0(\Gamma\backslash\mathfrak{H})$, with respect to norms

$$|f|_{M_0^r} = \sup_{\Im(z)\geq\sqrt{3}/2} y^{-r} \cdot |f(x+iy)|$$

Thus, $\lim_{y\to\infty} y^{-r}|f(x+iy)| = 0$ for $f \in M_0^r$. That is, the *set* M is $M = \bigcup_r M_0^r$, but the topology is perhaps not quite as expected. Nevertheless, M is a strict colimit (in the locally convex category) of Banach spaces, so is an LF-space, so is quasi-complete and locally convex.

6.4 Pre-trace formula and $\mathfrak{E}^{1+\varepsilon} \subset M$

From the Sobolev imbedding at the end of 3.5, we have $\mathfrak{E}^{1+\varepsilon} \subset C^0(\Gamma\backslash\mathfrak{H})$ for all $\varepsilon > 0$. Certainly $E_s \in C^0(\Gamma\backslash\mathfrak{H})$ away from poles, and $s \to E_s$ is a meromorphic $C^0(\Gamma\backslash\mathfrak{H})$-valued function. Thus, the Fréchet space $C^0(\Gamma\backslash\mathfrak{H})$ (with seminorms given by suprema on compact subsets) contains both $\mathfrak{E}^{1+\varepsilon}$ and Eisenstein series. However, the space M is much smaller, contains Eisenstein series, and $s \to E_s$ is a meromorphic M-valued function. Thus, we want

Lemma 39. $\mathfrak{E}^{1+\varepsilon} \subset M$*, for all* $\varepsilon > 0$.

Proof. A slightly more refined form of the pre-trace formula from 3.5 is

$$\sum_{|s_F| \le T} |F(z)|^2 + \frac{1}{4\pi} \int_{-T}^{T} |E_{\frac{1}{2}+it}(z)|^2 \, dt \quad \ll \quad T^2 + T \cdot \Im(z)$$

as $\Im(z) \to +\infty$, where F runs over an orthonormal basis for cuspforms, and F has eigenvalue $s_F(1-s_F)$ for the invariant Laplacian. Thus, certainly,

$$\frac{1}{4\pi} \int_{-T}^{T} |E_{\frac{1}{2}+it}(z)|^2 \, dt \quad \ll \quad T^2 + T \cdot \Im(z)$$

as $\Im z \to \infty$. As earlier, this asserts that the functional $\delta_z^{\rm nc}$ given by $\delta_z^{\rm nc} f = f(z)$ is in $\mathfrak{E}^{-1-\varepsilon}$, a weaker assertion than what we know to be true. However, the integration by parts does also yield an estimate for the $\mathfrak{E}^{-1-\varepsilon}$ norm depending on height $\Im(z)$, namely, $|\delta_z^{\rm nc}|_{\mathfrak{E}^{-1-\varepsilon}} \ll_\varepsilon y$ as $y \to \infty$, for all ε. For $u \in \mathfrak{E}^{1+\varepsilon}$, by the Cauchy-Schwarz-Bunyakowsky inequality extended to the pairing $\mathfrak{E}^{1+\varepsilon} \times \mathfrak{E}^{-1-\varepsilon} \to \mathbb{C}$,

$$|u_{\theta,w}(z)| = |\delta_z^{\rm nc} u_{\theta,w}| \le |u|_{\mathfrak{E}^{1+\varepsilon}} \cdot |\delta_z^{\rm nc}|_{\mathfrak{E}^{-1-\varepsilon}} \ll_\varepsilon |u|_{\mathfrak{E}^{1+\varepsilon}} \cdot \Im(z)$$

Thus, $\mathfrak{E}^{1+\varepsilon} \subset M_0^{1+\varepsilon} \subset M$ for every $\varepsilon > 0$. *Q.E.D.*

Therefore, the continuous dual M^* of M has a natural map to the duals $\mathfrak{E}^{-1-\varepsilon}$ of the spaces $\mathfrak{E}^{1+\varepsilon}$ for all $\varepsilon > 0$, removing a potential ambiguity:

Corollary 40. $\mathcal{E}\theta(s) = \theta E_{1-s}$ *for* $\theta = \overline{\theta} \in M^*$.

Proof. The proof of Proposition 9 applies here, with the Fréchet space $C^0(\Gamma\backslash\mathfrak{H})$ replaced by the (quasi-complete, locally convex) LF-space M. *Q.E.D.*

Using the latter, we have

Corollary 41. *The function* $s \to \mathcal{E}\theta(s) = \theta E_s$ *is a meromorphic function of* s.

Proof. From the previous corollary, indeed $\mathcal{E}\theta(s) = \theta E_{1-s}$. The function $s \to E_s$ is a meromorphic M-valued function, so $s \to \theta E_s$ is a meromorphic $\mathbb{C}$-valued function. *Q.E.D.*

6.5 Meromorphic continuation of spectral synthesis integrals

Consider vector-valued integrals

$$u_{\theta,w} = \frac{\theta(1)\cdot 1}{\lambda_1 - \lambda_w} + \frac{1}{4\pi i}\int_{(\frac{1}{2})} \frac{\mathcal{E}\theta(s)\cdot E_s}{\lambda_s - \lambda_w}\, d\, s$$

initially defined in $\Re(w) > \frac{1}{2}$ (and $w \neq 1$), where $\lambda_s = s(1-s)$, and where $\theta = \overline{\theta} \in \mathfrak{E}^{-1+\varepsilon}$ for some $\varepsilon > 0$. In that region, $u_{\theta,w}$ solves the differential equation $(-\Delta - \lambda_w)u = \theta$, and is in $\mathfrak{E}^{1+\varepsilon}$. More generally, let $\Phi : M \to N$ be a continuous linear map to a quasi-complete locally convex topological vector space N, and consider the N-valued integrals

$$u_{\theta,w,\Phi} = \frac{\theta(1)\cdot 1}{\lambda_1 - \lambda_w} + \frac{1}{4\pi i}\int_{(\frac{1}{2})} \frac{\mathcal{E}\theta(s)\cdot \Phi E_s}{\lambda_s - \lambda_w}\, d\, s$$

Of course, for Φ the identity map $M \to M$, this just gives $u_{\theta,w}$ itself. We anticipate that $\Phi(u_{\theta,w}) = u_{\theta,w,\Phi}$, but this needs proof.

From the previous section, $\theta \in \mathfrak{E}^{-1+\varepsilon} \subset \mathfrak{E}^{-1-\varepsilon}$ extends to an element of the dual M^* to the space M of moderate growth continuous functions. Thus, θ can be applied to Eisenstein series, and, further, $\mathcal{E}\theta(s) = \theta E_s$ by Corollary 40. Thus, the previous expression can be rewritten somewhat more concretely as

$$u_{\theta,w,\Phi} = \frac{\theta(1)\cdot 1}{\lambda_1 - \lambda_w} + \frac{1}{4\pi i}\int_{(\frac{1}{2})} \frac{\theta E_{1-s}\cdot \Phi E_s}{\lambda_s - \lambda_w}\, d\, s$$

Unsurprisingly, we have

Lemma 42. $\Phi(u_{\theta,w}) = u_{\theta,w,\Phi}$ *in the region* $\Re(w) > \frac{1}{2}$ *and* $w \neq 1$.

Proof. Again, in the region $\Re(w) > \frac{1}{2}$ and $w \neq 1$, the hypotheses guarantee, via the extended Plancherel theorem, that the integral for $u_{\theta,w}$ is an $\mathfrak{E}^{1+\varepsilon}$-valued holomorphic function of w. In that region, using properties of compactly supported, continuous-integrand Gelfand-Pettis integrals from 6.1,

$$\Phi\Big(\int_{(\frac{1}{2})} \frac{\theta E_{1-s}\cdot E_s}{\lambda_s-\lambda_w}\,\mathrm{d}s\Big) = \Phi\Big(\lim_{T\to+\infty}\int_{|\Im(s)|\le T} \frac{\theta E_{1-s}\cdot E_s}{\lambda_s-\lambda_w}\,\mathrm{d}s\Big)$$

$$= \lim_{T\to+\infty}\Phi\int_{|\Im(s)|\le T} \frac{\theta E_{1-s}\cdot E_s}{\lambda_s-\lambda_w}\,\mathrm{d}s = \lim_{T\to+\infty}\int_{|\Im(s)|\le T} \frac{\theta E_{1-s}\cdot \Phi E_s}{\lambda_s-\lambda_w}\,\mathrm{d}s$$

$$= \int_{(\frac{1}{2})} \frac{\theta E_{1-s}\cdot \Phi E_s}{\lambda_s-\lambda_w}\,\mathrm{d}s$$

since the limit is approached in $\mathfrak{E}^{1+\varepsilon}\subset M$. *Q.E.D.*

Further specific applications to $u_{\theta,w}$ will be presented later as corollaries to the following theorem.

Theorem 43. *With continuous linear $\Phi : M \to N$, with N quasi-complete and locally convex, the ΦM-valued function $w \to u_{\theta,w,\Phi}$ has a meromorphic continuation as an N-valued function of w. In further detail, the function*

$$J_{\theta,w,\Phi} = \frac{\theta(1)\cdot\Phi(1)}{(\lambda_1-\lambda_w)\cdot\langle 1,1\rangle} + \frac{1}{4\pi i}\int_{(\frac{1}{2})} \frac{\theta E_{1-s}\cdot\Phi E_s - \theta E_{1-w}\cdot\Phi E_w}{\lambda_s-\lambda_w}\,\mathrm{d}s$$

has a meromorphic continuation to an N-valued function with the functional equation $J_{\theta,1-w,\Phi} = J_{\theta,w,\Phi}$, and

$$u_{\theta,w,\Phi} = J_{\theta,w,\Phi} + \frac{\theta E_{1-w}\cdot\Phi E_w}{2(1-2w)}$$

Remark 44. *The continuation assertion for $u_{\theta,w}$ itself is stronger than, for example, the assertion of meromorphic continuation of the numerical, pointwise integrals*

$$u_{\theta,w}(z_o) = \frac{\theta(1)\cdot 1}{(\lambda_1-\lambda_w)\cdot\langle 1,1\rangle} + \frac{1}{4\pi i}\int_{(\frac{1}{2})} \frac{\theta E_{1-s}\cdot E_s(z_o)}{\lambda_s-\lambda_w}\,\mathrm{d}s$$

for fixed $z_o\in\mathfrak{H}$

Proof. From the lemma, in the region $\Re(w) > \frac{1}{2}$ and $w\neq 1$, the expression for $u_{\theta,w,\Phi}$ converges as an N-valued integral. The meromorphic continuation will be obtained through rearrangement of the integral. First, in $\Re(w) > \frac{1}{2}$ and $w \neq 1$, we can certainly add and subtract to obtain

$$u_{\theta,w,\Phi} = \frac{\theta(1)\cdot\Phi(1)}{(\lambda_1-\lambda_w)\cdot\langle 1,1\rangle} + \frac{1}{4\pi i}\int\limits_{(\frac{1}{2})} \frac{\theta E_{1-s}\cdot\Phi E_s}{\lambda_s-\lambda_w}\,\mathrm{d}s$$

$$= \frac{\theta(1)\cdot\Phi(1)}{(\lambda_1-\lambda_w)\cdot\langle 1,1\rangle} + \frac{1}{4\pi i}\int\limits_{(\frac{1}{2})} \frac{\theta E_{1-s}\cdot\Phi E_s - \theta E_{1-w}\cdot\Phi E_w}{\lambda_s-\lambda_w}\,\mathrm{d}s$$

$$+\theta E_{1-w}\cdot\Phi E_w\frac{1}{4\pi i}\int\limits_{(\frac{1}{2})}\frac{\mathrm{d}s}{\lambda_s-\lambda_w}$$

$$= J_{\theta,w,\Phi} + \theta E_{1-w}\cdot\Phi E_w\frac{1}{4\pi i}\int\limits_{(\frac{1}{2})}\frac{\mathrm{d}s}{\lambda_s-\lambda_w}$$

By residues,

$$\theta E_{1-w}\cdot\Phi E_w\frac{1}{4\pi i}\int\limits_{(\frac{1}{2})}\frac{\mathrm{d}s}{\lambda_s-\lambda_w} = \theta E_{1-w}\cdot\Phi E_w\Big(-\frac{1}{2}\cdot\mathrm{Res}_{s=w}\frac{1}{\lambda_s-\lambda_w}\Big)$$

$$= \frac{\theta E_{1-w}\cdot\Phi E_w}{2(1-2w)}$$

Since $w \to E_{1-w}$ is a meromorphic M-valued function and $\theta \in M^*$, the function $w \to \theta E_{1-w}$ is a meromorphic $\mathbb{C}$-valued function. Similarly, $w \to \Phi E_w$ is meromorphic N-valued. Thus, $\theta E_{1-w}\cdot\Phi E_w$ is a meromorphic N-valued function, with a meromorphic continuation from the meromorphic continuation of the Eisenstein series. Although the numerator is invariant under $w \to 1-w$ by the functional equation of the Eisenstein series, the denominator is skew-symmetric.

Now we meromorphically continue the integral $J_{\theta,w,\Phi}$. Constrain w to lie in a fixed compact set C, and take T large enough so that $T \geq 2|w|$ for all $w \in C$. At first for $\Re(w) > \frac{1}{2}$, make the obvious attempt to cancel vanishing of the denominator when s is close to w, by rearranging

$$J_{\theta,w,\Phi} - \frac{\theta(1)\cdot\Phi(1)}{(\lambda_1-\lambda_w)\cdot\langle 1,1\rangle} = \frac{1}{4\pi i}\int\limits_{(\frac{1}{2})} \frac{\theta E_{1-s}\cdot\Phi E_s - \theta E_{1-w}\cdot\Phi E_w}{\lambda_s-\lambda_w}\,\mathrm{d}s$$

$$= \frac{1}{4\pi i}\int\limits_{|t|\ge T} \frac{\theta E_{1-s}\cdot\Phi E_s}{\lambda_s-\lambda_w}\,\mathrm{d}s + \theta E_{1-w}\cdot\Phi E_w\cdot\frac{1}{4\pi i}\int\limits_{|t|\ge T}\frac{1}{\lambda_s-\lambda_w}\,\mathrm{d}s$$

$$+\frac{1}{4\pi i}\int\limits_{|t|\le T}\frac{\theta E_{1-s}\cdot\Phi E_s - \theta E_{1-w}\cdot\Phi E_w}{\lambda_s-\lambda_w}\,\mathrm{d}s$$

The meromorphy of the leading integral in the case of $u_{\theta,w}$ itself is understood via the Plancherel theorem on the continuous automorphic spectrum. Ignoring constants, the Plancherel theorem for $\mathfrak{E}^0$ asserts that, for $t\to A(t)$ in $L^2(\mathbb{R})$, the spectral synthesis integral

$$B(z) = \frac{1}{4\pi}\int_{-\infty}^{\infty} A(t)\cdot E_s(z)\,\mathrm{d}s$$

for $z\in\mathfrak{H}$ produces a function B in $\mathfrak{E}^0$, and the map $A\to B$ gives an isometry. Since L^2 functions in $\mathfrak{E}^0$ certainly need not have moderate pointwise growth at infinity, to have a continuous inclusion to M it is necessary to use $\mathfrak{E}^{1+\varepsilon}$. For $\mathfrak{E}^r$ for general index r, Plancherel becomes the following. Let X_r be the measurable functions $t\to A(t)$ (modulo null functions) on $\mathbb{R}$ such that $\int_{\mathbb{R}}|A(t)|^2\cdot(\frac{1}{4}+t^2)^r\,dt<\infty$. Then the spectral synthesis integral produces a function B in X_r, and $A\to B$ gives an isometry $\mathfrak{E}^r\to X_r$ (ignoring constants). Since $\theta\in\mathfrak{E}^{-1+\varepsilon}$, for w in a fixed compact,

$$\int_{|t|\ge T}\Big|\frac{\theta E_{\frac{1}{2}-it}}{\lambda_{\frac{1}{2}+it}-\lambda_w}\Big|^2\cdot\Big(\frac{1}{4}+t^2\Big)^{1+\varepsilon}\,\mathrm{d}t \;<\;\infty$$

Indeed, the $V^{1+\varepsilon_o}$-valued function that is $w\to(t\to\frac{\theta E_{\frac{1}{2}-it}}{\lambda_{\frac{1}{2}+it}-\lambda_w})$ for $|t|\ge T$, and is 0 for $|t|<T$, is directly seen to be complex differentiable M-valued in w, hence, holomorphic. Composition with the Plancherel isometry shows that

$$w\;\to\;\frac{1}{4\pi i}\int\limits_{|t|\ge T}\frac{\theta E_{1-s}\cdot E_s}{\lambda_s-\lambda_w}\,\mathrm{d}s$$

is a meromorphic $\mathfrak{E}^{1+\varepsilon}$-valued function in w in the fixed compact. Since $|w| \ll T$, the meromorphic continuation is given by the same integral, the invariance of the integrand under $w \to 1-w$ remains. This treats the first term for $u_{\theta,w}$.

To address the first term for $u_{\theta,w,\Phi}$ for continuous $\Phi : M \to N$, with inclusion $\mathfrak{E}^{1+\varepsilon} \subset M$, use properties of compactly supported, continuous-integrand Gelfand-Pettis integrals from 6.1:

$$\Phi\Big(\int_{|\Im(s)|\geq T} \frac{\theta E_{1-s}\cdot E_s}{\lambda_s-\lambda_w}\,\mathrm{d}s\Big) = \Phi\Big(\lim_{T'\to+\infty}\int_{T'\geq|\Im(s)|\geq T} \frac{\theta E_{1-s}\cdot E_s}{\lambda_s-\lambda_w}\,\mathrm{d}s\Big)$$

$$= \lim_{T'\to+\infty}\Phi\int_{T'\geq|\Im(s)|\geq T} \frac{\theta E_{1-s}\cdot E_s}{\lambda_s-\lambda_w}\,\mathrm{d}s = \lim_{T'\to+\infty}\int_{T'\geq|\Im(s)|\geq T} \frac{\theta E_{1-s}\cdot \Phi E_s}{\lambda_s-\lambda_w}\,\mathrm{d}s$$

$$= \int_{|\Im(s)|\geq T} \frac{\theta E_{1-s}\cdot \Phi E_s}{\lambda_s-\lambda_w}\,\mathrm{d}s$$

since the limit is approached in $\mathfrak{E}^{1+\varepsilon} \subset M$. Thus, the meromorphic continuation in $\mathfrak{E}^{1+\varepsilon}$ of the first term for $u_{\theta,w}$ gives that for $u_{\theta,w,\Phi}$.

In the second summand

$$\theta E_{1-w}\cdot\Phi E_w\cdot\frac{1}{4\pi i}\int\limits_{|t|\geq T}\frac{1}{\lambda_s-\lambda_w}\,\mathrm{d}s$$

the leading $\theta E_w\cdot\Phi E_w$ has a meromorphic continuation and is invariant under $w \to 1-w$. Since $|w| \ll T$, the meromorphic continuation of the integrand is given by the the same integral, and the invariance under $w \to 1-w$ remains.

The remaining summand

$$\frac{1}{4\pi i}\int\limits_{|t|\leq T}\frac{\theta E_{1-s}\cdot\Phi E_s - \theta E_{1-w}\cdot\Phi E_w}{\lambda_s-\lambda_w}\,\mathrm{d}s$$

is a compactly-supported vector-valued integral. To show that the integral is a meromorphic N-valued function of w, we invoke the Gelfand-Pettis criterion for existence of a weak integral. Let $\mathrm{Hol}(\Omega, N)$ be the topological vector space of holomorphic N-valued functions on a fixed open Ω avoiding poles of E_w, with compact closure C. It suffices to show that the integrand extends to a continuous $\mathrm{Hol}(\Omega, N)$-valued function of s, where $\mathrm{Hol}(\Omega, N)$ has the natural quasi-complete locally convex topology as in section 6.2. Unsurprisingly, to show that the integrand extends to a holomorphic (hence, continuous) $\mathrm{Hol}(\Omega, N)$-valued function of s, it suffices to show that the integrand extends to a holomorphic N-valued function of the two complex variables s, w.

By Cauchy-Goursat theory for vector-valued holomorphic functions, near a point s_o, the N-valued function $s \to \Phi E_s$ has a convergent power series expansion

$$\Phi E_s = A_0 + A_1(s - s_o) + A_2(s - s_o)^2 + \ldots$$

with $A_i \in N$, and θE_s has a scalar power series expansion

$$\theta E_s = \theta(A_0) + \theta(A_1) \cdot (s - s_o) + \theta(A_2) \cdot (s - s_o)^2 + \ldots$$

Thus, for suitable coefficients $B_n \in N$,

$$\theta E_{1-s} \cdot \Phi E_s = B_0 + B_1(s - s_o) + B_2(s - s_o)^2 + \ldots$$

Then

$$\begin{aligned}
&\theta E_{1-s} \cdot E_s - \theta E_{1-w} \cdot \Phi E_w \\
&\quad = B_1((s - s_o) - (w - s_o)) + B_2((s - s_o)^2 - (w - s_o)^2) + \ldots \\
&\quad = ((s - s_o) - (w - s_o)) \cdot (\text{convergent power series in } s - s_o,\, w - s_o) \\
&\quad = (s - w) \cdot (\text{convergent power series in } s - s_o,\, w - s_o)
\end{aligned}$$

Holomorphy is a local property. Thus, the integrand, initially defined only for $s \neq w$, extends to a holomorphic N-valued function $F(s, w)$ including the diagonal $s = w = \frac{1}{2} + it$ with $|t| \leq T$, as well. Thus, the $\mathrm{H}ol(\Omega, N)$-valued function $f(s)$ given by $f(s)(w) = F(s, w)$ is holomorphic in w. Thus, there is a Gelfand-Pettis integral $\int_{|t| \leq T} f(\frac{1}{2} + it)\, dt$ in $\mathrm{Hol}(\Omega, N)$, as desired. This proves the meromorphic continuation. Further, the $w \to 1 - w$ symmetry is retained by the extension of the integrand to the diagonal. *Q.E.D.*

Corollary 45. *For $\Phi : M \to N$ a continuous map from the space M of moderate-growth continuous functions on $\Gamma\backslash\mathfrak{H}$ to a quasi-complete, locally convex topological vector space N, the meromorphic continuations satisfy*

$$\Phi(u_{\theta,w}) = u_{\theta,w,\Phi}$$

Proof. In $\Re(w) > \frac{1}{2}$, lemma 42 proves the asserted equality. Then the vector-valued version of the identity principle of complex analysis gives equality of the meromorphic continuations. *Q.E.D.*

Corollary 46. *With* $\theta \in \mathfrak{E}^{-1+\varepsilon}$ *for some* $\varepsilon > 0$*, for* $\Re(w) < \frac{1}{2}$,

$$u_{\theta,w} = u_{\theta,1-w} + \frac{\theta E_w \cdot E_{1-w}}{1-2w}$$

In particular, for $\Re(w) < \frac{1}{2}$*,* $u_{\theta,w} \in \mathfrak{E}^{1+\varepsilon}$ *if and only if* $\theta E_{1-w} = 0$.

Proof. First, $\theta E_{1-w} \cdot E_w = \theta E_w \cdot E_{1-w}$, using the functional equations $E_{1-w} = E_w / c_w$ and $c_w c_{1-w} = 1$. Then, from the theorem, with $\Re(w) < \frac{1}{2}$,

$$u_{\theta,w} = J_{\theta,w} + \frac{\theta E_{1-w} \cdot E_w}{2(1-2w)} = J_{\theta,1-w} - \frac{\theta E_w \cdot E_{1-w}}{2(1-2(1-w))}$$

$$= \Big(J_{\theta,1-w} + \frac{\theta E_w \cdot E_{1-w}}{2(1-2(1-w))} \Big) - 2\frac{\theta E_w \cdot E_{1-w}}{2(1-2(1-w))} = u_{\theta,1-w} - 2\frac{\theta E_w \cdot E_{1-w}}{2(1-2(1-w))}$$

Since $\Re(1-w) > \frac{1}{2}$, $u_{\theta,1-w} \in \mathfrak{E}^{1+\varepsilon}$, so $u_{\theta,w} \in \mathfrak{E}^{1+\varepsilon}$ if and only if the extra summand vanishes. *Q.E.D.*

Again, taking $\Phi = \theta$ in the above, let

$$J_{\theta,w} = J_{\theta,w,\theta} = \frac{\theta(1) \cdot 1}{(\lambda_1 - \lambda_w) \cdot \langle 1, 1 \rangle} + \frac{1}{4\pi i} \int_{(\frac{1}{2})} \frac{\theta E_{1-s} \cdot \theta E_s - \theta E_{1-w} \cdot \theta E_w}{\lambda_s - \lambda_w} \, ds$$

Corollary 47. *For* $\Re(w) = \frac{1}{2}$*,* $\Re \theta u_{\theta,w} = J_{\theta,w}$ *and* $\Im \theta u_{\theta,w} = -\theta E_{1-w} \cdot \theta E_w / 4\Im(w)$.

Proof. In Theorem 43, with $\Phi = \theta$, on $\Re(w) = \frac{1}{2}$, complex conjugation applied to (the meromorphic continuation of the) the integral for $J_{\theta,w,\theta}$ sends $w \to 1-w$, under which $J_{\theta,w,\theta}$ is invariant, since $\overline{\theta} = \theta$. On the other hand, by the functional equation of E_w, the term $\theta E_{1-w} \cdot \theta E_w / 2(1-2w)$ is purely imaginary. *Q.E.D.*

6.6 Spectral corollaries of meromorphic continuation

Corollary 48. *With* $\theta \in \mathfrak{E}^{-1+\varepsilon}$ *for some* $\varepsilon > 0$*, at points* w *with* $\Re(w) \leq \frac{1}{2}$*, the meromorphic continuation of* $u_{\theta,w}$ *as* M*-valued function of* w *is in* $\mathfrak{E}^{1+\varepsilon}$ *if and only if* $\theta E_w = 0$.

Proof. The theorem 43 with Φ the identity map of M to itself gives the meromorphic continuation of $u_{\theta,w}$ as an M-valued function. For $\theta E_w = 0$, the extra term involving Eisenstein series in the expression

$$u_{\theta,w} = J_{\theta,w} + \frac{\theta E_{1-w} \cdot E_w}{2(1-2w)}$$

disappears. Further, in the numerator in the integral

$$J_{\theta,w} = \frac{\theta(1) \cdot 1}{\lambda_1 - \lambda_w} + \frac{1}{4\pi i} \int_{(\frac{1}{2})} \frac{\theta E_{1-s} \cdot E_s - \theta E_{1-w} \cdot E_w}{\lambda_s - \lambda_w} \, \mathrm{d}s$$

the term θE_w vanishes. Thus, the spectral coefficient $s \to \theta E_{1-s}/(\lambda_s - \lambda_w)$ in the integrand is in $X_{1+\varepsilon}$. By the extended Plancherel theorem, the integral (extended by continuity) is in $\mathfrak{E}^{1+\varepsilon}$. *Q.E.D.*

Corollary 49. *With $\theta \in \mathfrak{E}^{-1+\varepsilon}$ for some $\varepsilon > 0$, if the Friedrichs extension $\widetilde{S}_\theta$ has an eigenfunction u with eigenvalue $\lambda_w > \frac{1}{4}$, then u is a multiple of the meromorphic continuation of $u_{\theta,w}$ to $\Re(w) = \frac{1}{2}$. Further, $\theta E_w = 0 = \theta E_{1-w}$ and $\theta u_{\theta,w} = 0$. Conversely, if $\theta E_w = 0 = \theta E_{1-w}$ and $\theta u_{\theta,w} = 0$, then $u_{\theta,w}$ is an eigenfunction of $\widetilde{S}_\theta$.*

Proof. From theorem 11, if $\lambda_w > \frac{1}{4}$ is an eigenvalue for $\widetilde{S}_\theta$, then $\theta E_w = 0$, and $\Re(w) = \frac{1}{2}$. Then, also, $\theta E_{1-w} = 0$ by the functional equation of the Eisenstein series. Thus, $u_{\theta,w} \in \mathfrak{E}^{1+\varepsilon}$, by the previous. Then for some constant c, $v = u - c \cdot u_{\theta,w}$ satisfies the homogeneous equation $(-\Delta - \lambda_w)v = 0$, which has no non-zero solution in $\mathfrak{E}^1$. For $u \in \mathfrak{E}^1$ to be in the domain of the Friedrichs extension, it is necessary and sufficient that $\theta u = 0$. *Q.E.D.*

As a special case of the theorem 43 with $\Phi = \theta : M \to \mathbb{C}$, whose relevance is clearer in light of the two previous corollaries, we have

Corollary 50. *The condition $\theta u_{\theta,w} = 0$ on the meromorphically continued $u_{\theta,w}$ is $u_{\theta,w,\theta} = 0$ with the meromorphic continuation of $u_{\theta,w,\theta}$ as in theorem 43.*

Remark 51. *If we did not know that any solution $u \in \mathfrak{E}^{1+\varepsilon}$ to the differential equation were of the form $u_{\theta,w}$, it would be more difficult to appraise the vanishing requirement $\theta u = 0$.*

Corollary 52. *A necessary and sufficient condition for $\lambda_w > \frac{1}{4}$ to be an eigenvalue for $\widetilde{S}_\theta$ is that $\theta u_{\theta,w} = 0$.*

Proof. By corollary 49, in that range, the eigenfunctions for $\widetilde{S}_\theta$ are exactly $u_{\theta,w}$ with $\theta E_w = 0$ and $\theta u_{\theta,w} = 0$. As above,

$$\theta u_{\theta,\theta} = \frac{\theta(1)\cdot\theta(1)}{(\lambda_1-\lambda_w)\cdot\langle 1,1\rangle} + \frac{1}{4\pi i}\int\limits_{(\frac{1}{2})} \frac{\theta E_{1-s}\cdot\theta E_s - \theta E_{1-w}\cdot\theta E_w}{\lambda_s-\lambda_w}\,\mathrm{d}s + \frac{\theta E_{1-w}\cdot\theta E_w}{2(1-2w)}$$

On $\Re(w) = \frac{1}{2}$, the first two summands are real, while the third is purely imaginary. Thus, for this to vanish on $\Re(w) = \frac{1}{2}$ requires vanishing of θE_w. That is, on $\Re(w) = \frac{1}{2}$, vanishing of $\theta u_{\theta,w}$ implies that of θE_w. *Q.E.D.*

For example, the self-adjoint operator $\widetilde{S}_\Theta$ on non-cuspidal pseudo-cuspforms, has purely discrete spectrum, by Theorem 16. Theorem 18 already noted that, for $\Re(w) = \frac{1}{2}$, $\lambda_w = w(1-w)$ is an eigenvalue if and only if $a^w + c_w a^{1-w} = 0$. Indeed, $\lambda_w = w(1-w)$ is real if and only if $\Re(w) = \frac{1}{2}$ or $w \in [0,1]$. However, this conclusion does not address the possible vanishing of $a^w + c_w a^{1-w}$ *off* $\Re(w) = \frac{1}{2}$ and $[0,1]$. As an instance of desirable corollaries of spectral theory, we have a result already observed in [17], but there for non-spectral reasons:

Corollary 53. $a^w + c_w a^{1-w} = 0$ *implies* $\Re(w) = \frac{1}{2}$ *or* $w \in [0,1]$.

Proof. Let $u_{\eta_a,w}$ be the meromorphic continuation of the spectral expansion

$$u_{\eta_a,w} = \frac{\eta_a(1)\cdot 1}{(\lambda_1-\lambda_w)\cdot\langle 1,1\rangle} + \frac{1}{4\pi i}\int_{(\frac{1}{2})} \frac{\eta_a E_{1-s}\cdot E_s}{\lambda_s-\lambda_w}\,\mathrm{d}s$$

By Corollary 49, $u_{\eta_a,w}$ gives an eigenfunction of $\widetilde{S}_\Theta$ if and only if $\eta_a E_w = 0$ and $\eta_a u_{\eta_a,w} = 0$. Of course, $\eta_a E_w = a^w + c_w a^{1-w}$. The computation of Theorem 24 gives

$$\eta_a u_{\eta_a,w} = \frac{a^{1-w}}{1-2w}\cdot(a^w + c_w a^{1-w})$$

The leading factor $a^{1-w}/(1-2w)$ does not vanish, and does not have poles except at $w = \frac{1}{2}$. Thus, if $a^w + c_w a^{1-w} = 0$, then $u_{\eta_a,w}$ is an eigenfunction for $\widetilde{S}_\Theta$, and necessarily λ_w is real and non-negative. Thus, for w outside the set where $\Re(w) = \frac{1}{2}$ or $w \in [0,1]$, the expression $a^w + c_w a^{1-w}$ cannot vanish. *Q.E.D.*

6.7 Remarks on appealing but incorrect arguments

The invariance of λ_w under $w \longleftrightarrow 1-w$ might suggest invariance of $u_{\theta,w}$ under $w \longleftrightarrow 1-w$. However, this is illusory, by corollary 46. Nevertheless, the implications of (the generally incorrect claim) $u_{\theta,1-w} = u_{\theta,w}$ are striking. That is, if we were to believe that $u_{\theta,1-w} = u_{\theta,w}$, the the trivial non-vanishing of remark 14 would (fairly provocatively, but incorrectly, seem to) show that $\theta u_{\theta,w} \neq 0$ off $\Re(w) = \frac{1}{2}$ and $[0,1]$.

Corollary 52 (correctly) shows that for $\lambda_w > \frac{1}{4}$, that is, for $\Re(w) = \frac{1}{2}$ and $w \neq \frac{1}{2}$, a necessary and sufficient condition for λ_w to be an eigenvalue for the self-adjoint operator $\widetilde{S}_\theta$ is that $\theta u_{\theta,w} = 0$. An argument principle discussion (going back to [1], see [27] pp. 212-213) shows that the asymptotic count of the zeros w of $\theta u_{\theta,w}$ with imaginary parts $0 \leq \Im(w) \leq T$ is $\frac{T}{\pi} \log T + O(T)$.

Thus, if we (erroneously) believe the confinement of zeros of $\theta u_{\theta,w}$ to $\Re(w) = \frac{1}{2}$ (and $[0,1]$), we (seem to) find that $\theta u_{\theta,w}$ asymptotically has $\frac{T}{\pi} \log T + O(T)$ zeros from height 0 to T on the critical line. Since $\theta u_{\theta,w} = 0$ on the critical line does truly entail $\theta E_w = 0$, we (seem to) conclude that the asymptotic count of zeros of θE_w on is $\frac{T}{\pi} \log T + O(T)$. For θ the Eisenstein-Heegner distribution attached to a fundamental discriminant $-d < 0$ and $k = \mathbb{Q}(\sqrt{-d})$, this would (seem to) imply that $\zeta_k(s)$ has 100% of its zeros on the critical line, the asymptotic version of the Riemann Hypothesis for $\zeta_k(s)$. The argument is fundamentally flawed, at the point where one fallaciously imagines that $u_{\theta,1-w} = u_{\theta,w}$: the functional equation of $u_{\theta,w}$ involves an extra term, which vanishes exactly for $\theta E_w = 0$, thus thwarting this naively optimistic approach.

7 Spacing of spectral parameters

7.1 Exotic eigenfunction expansions of distributions

Fix $a > 1$. Let $\{f_j : j = 1, 2, \ldots\}$ be eigenfunctions for the operator $\widetilde{S}_\Theta$ of section 4.2, with eigenvalues $\lambda_{s_1} \leq \lambda_{s_2} \leq \ldots$, with $|f_j|_V = 1$. As in Corollary 21, $|f_j|_{\mathfrak{E}^1} = \sqrt{\lambda_{s_j}}$. Let $i_\Theta : \mathfrak{E}^1 \cap \mathfrak{E}^0_\Theta \to \mathfrak{E}^1$ be the inclusion, and $i^*_\Theta : \mathfrak{E}^{-1} \to (\mathfrak{E}^1 \cap \mathfrak{E}^0_\Theta)^*$ its adjoint. By Theorem 20 , $\mathfrak{E}^1 \cap \mathfrak{E}^0_\Theta$ is the $\mathfrak{E}^1$-topology completion of $D \cap \mathfrak{E}^0_\Theta$, where D is the space of pseudo-Eisenstein series with test-function data. Thus, $\{f_j\}$ is an orthonormal basis for $\mathfrak{E}^0_\Theta$, and $\{f_j/\sqrt{\lambda_{s_j}}\}$ is an orthonormal basis for

$\mathfrak{E}^1 \cap \mathfrak{E}^0_\Theta$.

At first for finite sums, define Sobolev-like norms by

$$\Big| \sum_j c_j \cdot f_j \Big|^2_{W_r} = \sum_j |c_j|^2 \cdot \lambda^r_{s_j}$$

and let W_r be the completion of the space of finite linear combinations of the vectors f_j with respect to this norm. Thus, $\{f_j \cdot \lambda_{s_j}^{-r/2}\}$ is an orthonormal basis for W_r.

Theorem 54. *The map $W_{-1} \approx (\mathfrak{E}^1 \cap \mathfrak{E}^0_\Theta)^*$ given by (at first for finite sums, and then extending by continuity)*

$$\Big(\sum_i a_i \cdot f_i\Big)\Big(\sum_j b_j \cdot f_j\Big) = \sum_j a_i \cdot \overline{b_j}$$

for $\sum_j b_j \cdot f_j \in \mathfrak{E}^1 \cap \mathfrak{E}^0_\Theta$ is an isomorphism.

Proof. First, since $f_j \in \mathfrak{E}^1$ by the Friedrichs construction, and $f_j \in \mathfrak{E}^0_\Theta$, it makes sense to apply such functionals in $\mathfrak{E}^{-1}$ to the eigenfunctions f_j. By Plancherel and Corollary 21, $W_0 = \mathfrak{E}^0_\Theta$ and $W_1 = \mathfrak{E}^1 \cap \mathfrak{E}^0_\Theta$. Generally, the natural pairing

$$\langle \sum_j a_j \cdot f_j, \sum_j b_j \cdot f_j \rangle = \sum_j a_j \, \overline{b_j}$$

on $W_r \times W_{-r}$ puts these two spaces in duality. Thus, we have a natural isomorphism $W_{-1} \to W_1^* = (\mathfrak{E}^1 \cap \mathfrak{E}^0_\Theta)^*$ respecting the duality pairings. *Q.E.D.*

Thus, functionals $\mu \in (\mathfrak{E}^1 \cap \mathfrak{E}^0_\Theta)^*$ admit expansions

$$\mu = \sum_j \mu(\overline{f}_j) \cdot f_j$$

convergent in the W_{-1} topology, and compatible with evaluation on the corresponding expansions of elements in $W_1 = \mathfrak{E}^1 \cap \mathfrak{E}^0_\Theta$.

7.2 Exotic eigenfunction expressions for solutions of equations

Let $\{f_j : j = 1, 2, \ldots\}$ be eigenfunctions for the operator $\widetilde{S}_\Theta$ of section 4.2, with eigenvalues $\lambda_{s_1} \leq \lambda_{s_2} \leq \ldots$, with $|f_j|_V = 1$.

Proposition 55. *The operator $S_\Theta^\#$ of Lemma 7 is a topological isomorphism from $W_1 = \mathfrak{E}^1 \cap \mathfrak{E}^0_\Theta$ to $W_{-1} = (\mathfrak{E}^1 \cap \mathfrak{E}^0_\Theta)^*$, expressible as*

$$S_\Theta^\# \sum_j b_j \cdot f_j = \sum_j b_j \cdot S_\Theta^\# f_j = \sum_j \lambda_{s_j} \cdot b_j \cdot f_j$$

Proof. Since constants are excluded from $\mathfrak{E}^0_\Theta$, S_Θ is *strictly* positive, so we are in the situation of section 2.2, and its Friedrichs extension gives the asserted isomorphism, as observed just prior to Proposition 1. Because $S_\Theta^\#$ is continuous $W_1 \to W_{-1}$, it commutes with limits, giving the formula. *Q.E.D.*

Theorem 56. *For $\mu \in W_{-1} = (\mathfrak{E}^1 \cap \mathfrak{E}^0_\Theta)^*$, for $\lambda_w \notin [0, +\infty)$, the equation $(\widetilde{S}_\Theta - \lambda_w)v = \mu$ has a unique solution in $W_1 = \mathfrak{E}^1 \cap \mathfrak{E}^0_\Theta$, given by*

$$v = \sum_j \frac{\mu(\overline{f}_j) \cdot f_j}{\lambda_{s_j} - \lambda_w}$$

This expansion is holomorphic W_1-valued off $\Re(w) = \frac{1}{2}$, and has a W_1-valued analytic continuation to $\mathbb{C}$ with the discrete set of spectral parameters s_j removed.

Proof. First, by the construction of the Friedrichs extension, all eigenfunctions f_j are in $\mathfrak{E}^1$, so any $\theta \in \mathfrak{E}^{-1}$ can be sensibly applied to them via the $\mathfrak{E}^1 \times \mathfrak{E}^{-1}$ pairing. The bound

$$\sum_j |\mu(\overline{f}_k)|^2 \cdot \lambda_{s_j}^{-1} < \infty$$

implies

$$\sum_j \Big| \frac{\mu(\overline{f}_j)}{\lambda_{s_j} - \lambda_w} \Big|^2 \cdot \lambda_{s_j} < \infty$$

so the given expansion for v is indeed in W_1. Since $S_\Theta^\# : W_1 \to W_{-1}$ is continuous, it commutes with the implied limits in the infinite sums.

To show the holomorphy, from section 6.2 it suffices to prove that for w in a fixed compact C not meeting $\mathbb{R}$, for every $\varepsilon > 0$ there is i_o such that for all $i_1, i_2 \geq i_o$

$$\sup_{w \in C} \Big| \sum_{i_1 \leq j \leq i_2} \frac{\mu(\overline{f}_j)}{\lambda_{s_j} - \lambda_w} \Big|_{W_1} < \varepsilon$$

This follows from $|\lambda_{s_j} - \lambda_w| \gg_C |\lambda_{s_j}|$. By Theorem 16, $\widetilde{S}_\Theta$ has compact resolvent, so the parameters s_j have no accumulation point in $\mathbb{C}$. Thus, the same inequality holds for C compact not meeting the set of spectral parameters s_j, giving the analytic continuation. *Q.E.D.*

Let $\theta = \overline{\theta} = \theta_{\mathcal{D}} = \sum_d \nu_d \, \theta_d$ be a finite real-linear combination of Heegner distributions θ_d, as in 3.6, with fundamental discriminants $d < 0$. As noted in Section 3.6, $\theta \in \mathfrak{E}^{-1+\varepsilon}$ for some $\varepsilon > 0$. Let $m(\mathcal{D}) = \max_d |d|^{\frac{1}{2}}/2$ be the maximum over d appearing with non-zero coefficient, and increase a if necessary so that $a > 1$.

Corollary 57. *The equation $(S_\Theta^\# - \lambda_w)v = i_\Theta^* \theta$ has unique solution*

$$v_{\theta,w} = \sum_j \frac{\theta(\overline{f}_j)}{\lambda_{s_j} - \lambda_s} \cdot f_j$$

in $W_1 = \mathfrak{E}^1 \cap \mathfrak{E}^0_\Theta$, a meromorphic W_1-valued function of w away from the spectral parameters s_j, and

$$\theta v_{\theta,w} = \sum_j \frac{|\theta f|^2}{\lambda_{s_j} - \lambda_s} \cdot f_j$$

is uniformly absolutely convergent on compacts away from the spectral parameters s_j, producing a holomorphic $\mathbb{C}$-valued function there.

Proof. The image $i_\Theta^* \theta$ is in W_{-1}. *Q.E.D.*

Corollary 58. *A solution $u \in \mathfrak{E}^1$ to the equation $(-\Delta - \lambda_w)u = \theta$ is in fact in $\mathfrak{E}^1 \cap \mathfrak{E}^0_\Theta$, so is $v_{\theta,w}$ of the previous corollary, so has an expansion*

$$u = \sum_j \frac{\theta(\overline{f}_j)}{\lambda_{s_j} - \lambda_w} \cdot f_j$$

convergent in $\mathfrak{E}^1 \cap \mathfrak{E}^0_\Theta$, holomorphic in w away from spectral parameters s_j, and there

$$\theta u = \sum_j \frac{|\theta(f_j)|^2}{\lambda_{s_j} - \lambda_w}$$

Proof. First, we recall why such a solution u must be of the form $u_{\theta,w}$. Certainly in $\Re(w) > \frac{1}{2}$ (and $w \neq 1$) that equation has the solution given by a spectral expansion converging in $\mathfrak{E}^1$:

$$u_{\theta,w} = \frac{\theta(1)\cdot 1}{(\lambda_1-\lambda_w)\cdot\langle 1,1\rangle} + \frac{1}{4\pi i}\int_{(\frac{1}{2})} \frac{\theta E_{1-s}\cdot E_s}{\lambda_s-\lambda_w}\, \mathrm{d}s$$

On $\Re(s)=\frac{1}{2}$, Theorem 10 shows that the equation is solvable only if $\mathcal{E}\theta(w)=0$, and Corollary 40 gives $\theta E_{1-w}=\mathcal{E}\theta(w)=0$. Thus, by Theorem 48, at such w the meromorphic continuation of $u_{\theta,w}$ is in $\mathfrak{E}^1$. Since the homogeneous form of the equation has no solution in $\mathfrak{E}^1$, it must be that $u=u_{\theta,w}$. From Theorem 26, the condition $\theta E_w=0$ gives $u_{\theta,w}\in\mathfrak{E}^1\cap\mathfrak{E}^0_\Theta$ for $a\geq m(\mathcal{D})$ (and $a>1$). The previous corollary applies to the image $\mu=i^*_\Theta\theta$, noting that for $f\in\mathfrak{E}^1\cap\mathfrak{E}^0_\Theta$,

$$(i^*_\Theta\theta)(f)=\theta(i_\Theta f)=\theta(f)$$

suppressing explicit reference to the inclusion i_Θ. Then $u_{\theta,w}$ must be the solution $v_{\theta,w}$ of the previous corollary. *Q.E.D.*

7.3 Interleaving

Continue to let $\theta=\overline{\theta}=\theta_{\mathcal{D}}=\sum_d \nu_d\,\theta_d$ be a finite real-linear combination of Eisenstein-Heegner distributions θ_d, as in 3.6, so that all the previous results apply. Let

$$v_{\theta,w}=\sum_j \frac{\theta(\overline{f}_j)}{\lambda_{s_j}-\lambda_s}\cdot f_j$$

be as in Corollary 57.

Corollary 59. *Let $s_j=\frac{1}{2}+it_j$ and $s_{j+1}=\frac{1}{2}+it_{j+1}$ be two adjacent zeros of $a^s+c_s a^{1-s}$ on the line $\Re(s)=\frac{1}{2}$, with $0<t_j<t_{j+1}$. For $w=\frac{1}{2}+i\tau$ with $t_j<\tau<t_{j+1}$, the function $\tau\to\theta v_{\theta,w}$ is continuous, real-valued, strictly increasing, goes from $-\infty$ as $\tau\to t_j$ to $+\infty$ as $\tau\to t_{j+1}$.*

Proof. For $\tau\in\mathbb{R}$,

$$\theta v_{\theta,\frac{1}{2}+i\tau}=\sum_j \frac{|\theta(f_j)|^2}{t_j^2-\tau^2}$$

is real-valued (away from the t_j). Being the restriction of a meromorphic function, $\tau\to\theta v_{\theta,\frac{1}{2}+i\tau}$ is continuous away from the t_j. On one hand, for $\tau\in(t_j,t_{j+1})$, as $\tau\to t_j^+$, the summand $|\theta(f_j)|^2/(t_j^2-\tau^2)$ goes to $-\infty$ and the rest of the sum remains finite. On the other hand, in that interval, as $\tau \to t_{j+1}^-$, the summand $|\theta(f_{j+1})|^2/(t_{j+1}^2 - \tau^2)$ goes to $+\infty$ and the rest of the the sum remains finite. Away from the poles, the derivative is

$$\frac{\partial}{\partial \tau}\theta v_{\theta,\frac{1}{2}+i\tau} = \frac{\partial}{\partial \tau}\sum_j \frac{|\theta(f_j)|^2}{t_j^2 - \tau^2} = 2\tau \cdot \sum_j \frac{|\theta(f_j)|^2}{(t_j^2 - \tau^2)^2} > 0$$

so the function is strictly increasing. *Q.E.D.*

The first corollary directly addresses on-the-line zeros of θE_w:

Corollary 60. *Let $s_j = \frac{1}{2}+it_j$ and $s_{j+1} = \frac{1}{2}+it_{j+1}$ be adjacent zeros of $a^s+c_sa^{1-s}$ on the line $\Re(s) = \frac{1}{2}$, with $0 < t_j < t_{j+1}$. Let $w_1 = \frac{1}{2} + i\tau_1$ and $w_2 = \frac{1}{2} + i\tau_2$ with $t_j < \tau_1 < \tau_2 < t_{j+1}$ with $\theta E_{w_j} = 0$. Then $\theta u_{\theta,w_1} < \theta u_{\theta,w_2}$.*

Proof. As above, the condition $\theta E_w = 0$ implies that the meromorphic continuation of $u_{\theta,w}$ is in $\mathfrak{E}^1$, and also that $u_{\theta,w} \in \mathfrak{E}^0_\Theta$. By Corollary 57, $u_{\theta,w}$ must be $v_{\theta,w}$, and $\theta v_{\theta,w}$ is monotone in intervals (t_j, t_{j+1}). *Q.E.D.*

This second corollary is a variant that more directly addresses solutions of the equation $(-\Delta - \lambda_w)u = \theta$:

Corollary 61. *Let s_j and s_{j+1} be adjacent zeros of $a^s + c_sa^{1-s}$ on the line $\Re(s) = \frac{1}{2}$. On the line $\Re(w) = \frac{1}{2}$, between s_j and s_{j+1}, there is at most one w such that a solution $u \in \mathfrak{E}^1$ of the equation $(-\Delta - \lambda_w)u = \theta$ satisfies $\theta u = 0$.*

Proof. Again, if a solution u to $(-\Delta - \lambda_w)u = \theta$ is in $\mathfrak{E}^1$, then $\theta E_w = 0$, by Theorem 10. And, again, the meromorphically-continued $u_{\theta,w}$ is in $\mathfrak{E}^1$ at that point, and is the unique solution to the equation, since the homogeneous equation $(-\Delta - \lambda_w)u = 0$ has no non-zero solution. Also, again, $\theta E_w = 0$ implies that $u_{\theta,w} \in \mathfrak{E}^0_\Theta$. Thus, by Corollary 58, $u_{\theta,w} = v_{\theta,v}$, and then

$$\theta u_{\theta,w} = \theta v_{\theta,w} = \sum_j \frac{|\theta(f_j)|^2}{\lambda_{s_j} - \lambda_w}$$

The monotonicity assertion of Corollary 59 shows that there is exactly one $w = \frac{1}{2} + i\tau$ in the given interval such that $\theta v_{\theta,w} = 0$. *Q.E.D.*

Corollary 62. *There is at most one w on $\Re(w) = \frac{1}{2}$ between adjacent zeros s_j and s_{j+1} of $a^s + c_sa^{1-s}$ on $\Re(s) = \frac{1}{2}$ such that λ_w is an eigenvalue for $\widetilde{S}_\theta$.*

Proof. Combine the previous corollary with Corollary 52. *Q.E.D.*

7.4 Spacing of spectral parameters

Continue to let $\theta = \overline{\theta} = \theta_{\mathcal{D}} = \sum_d \nu_d \, \theta_d$ be a finite real-linear combination of Eisenstein-Heegner distributions θ_d, as in 3.6, so that the immediately previous results apply.

From the previous section, the positions of the zeros of $a^s + c_s a^{1-s}$ influence the positions of the parameters w appearing in differential equations $(-\Delta - \lambda_w)u = \theta$ with condition $\theta u = c \in \mathbb{R}$. The zeros s of $a^s + c_s a^{1-s} = 0$ allow a degree of adjustment by choice of the cut-off height $a > 1$.

To anticipate the point of the present discussion, for $\log\log \Im(s)$ large, the behavior of $\zeta(s)$ on the edge of the critical strip is relatively regular, by [27] (5.17.4) page 112 (in an earlier edition, page 98). This gives an eventual regularity of spacing of the zeros of $a^s + c_s a^{1-s}$, with implications for the exotic eigenfunction expansions above.

As originally in [1], or from [27] pages 212–213, by the argument principle and Jensen's inequality,

$$\#\Big(\text{zeros of } a^s + c_s a^{1-s} \text{ with } 0 \le \Im(s) \le T\Big) \quad = \quad \frac{T}{\pi} \log T + O(T)$$

Thus, near height T, the average gap is $\pi/\log T$. As recalled in Corollary 53, all the zeros are on $\frac{1}{2} + i\mathbb{R}$ or $[0,1]$, and are in bijection with the discrete spectrum of $\widetilde{S}_\Theta$ (modulo $w \leftrightarrow 1-s$) by $s \to \lambda_s = s(1-s)$.

First, $|c_s| = 1$ on $\Re(s) = \frac{1}{2}$, so $c_{\frac{1}{2}+it} = e^{-2i\psi(t)}$ with real-valued $\psi(t) = \arg \xi(1+2it)$. Since $\zeta(s)$ does not vanish on $\Re(s) = 1$, $\psi(t)$ is differentiable. Letting $s = \frac{1}{2} = it$, rearrange

$$\begin{aligned} a^s + c_s a^{1-s} &= \sqrt{a} \cdot e^{-i\psi(t)} \cdot \Big(e^{it\log a + i\psi(t)} + e^{-it\log a - i\psi(t)}\Big) \\ &= 2\sqrt{a} \cdot e^{-i\psi(t)} \cdot \cos\Big(t\log a + \psi(t)\Big) \end{aligned}$$

Thus, the on-the-line vanishing condition is $\cos(t\log a + \psi(t)) = 0$.

Proposition 63. $\psi(t) = \arg \xi(1+2it)$ *satisfies*

$$\psi(t) = t\log t + O(\frac{t\log t}{\log\log t}) \qquad \textit{and} \qquad \psi'(t) = \log t + O(\frac{\log t}{\log\log t})$$

Proof. Of course,

$$\psi(t) = \arg\xi(1+2it) = -t\log\pi + \arg\Gamma(\frac{1}{2}+it) + \arg\zeta(1+2it)$$

From $\log\Gamma(s) = (s-\frac{1}{2})\log s - s + \frac{1}{2}\log 2\pi + O(s^{-1})$

$$\log\Gamma(\frac{1}{2}+it) = it\log(1+it) - (\frac{1}{2}+it) + \frac{1}{2}\log 2\pi + O\Big(1\frac{1}{2}+it\Big)$$
$$= it\Big(\log t + O(\frac{1}{t^2}) + i(\frac{\pi}{2}+O(\frac{1}{t}))\Big) - (\frac{1}{2}+it) + \frac{1}{2}\log 2\pi + O\Big(\frac{1}{\frac{1}{2}+it}\Big)$$

Thus,

$$\Im\log\Gamma(\frac{1}{2}+it) \;=\; t\log t - t + O(\frac{1}{t})$$

and

$$\psi(t) \;=\; (-t\log\pi) + (t\log t - t) + \arg\zeta(1+2it) + O(t^{-1})$$

Similarly, the asymptotic $\Gamma'(s)/\Gamma(s) = \log s + O(s^{-1})$ gives

$$\Im\frac{d}{ds}\log\Gamma(\frac{1}{2}+it) \;=\; \log t + O(t^{-1})$$

and

$$\psi'(t) \;=\; (-\log\pi) + (\log t) + \frac{d}{dt}\arg\zeta(1+2it) + O(t^{-1})$$

From [27] (5.17.4) page 112 (in an earlier edition, page 98), for $u \geq t$,

$$\log\zeta(1+iu) - \log\zeta(1+it) \;=\; O\Big(\frac{\log t}{\log\log t}\Big)\cdot(u-t)$$

This gives the asymptotics for ψ and ψ'. *Q.E.D.*

Corollary 64. *Fix $a_o > 1$. Given $\varepsilon > 0$ (with $\varepsilon < 1$ for definiteness), for sufficiently large $T_o > 0$, for all $1 < a \leq a_o$, for consecutive real zeros $t_j < t_{j+1}$ of $a^{\frac{1}{2}+it} + c_{\frac{1}{2}+it}a^{\frac{1}{2}-it}$ with $T_o \leq t_j < t_{j+1}$,*

$$(1-\varepsilon)\cdot\frac{\pi}{\log t_j} \leq t_{j+1} - t_j \leq (1+\varepsilon)\frac{\pi}{\log t_j}$$

Proof. This is elementary from the previous. *Q.E.D.*

Theorem 65. *The real zeros* $t = t(a)$ *of* $a^{\frac{1}{2}+it} + c_{\frac{1}{2}+it} a^{\frac{1}{2}-it}$*, which are also the real zeros of* $\cos(t \log a + \psi(t))$*, are differentiable functions of cut-off height* a*, with*

$$\frac{\partial t}{\partial a} = \frac{-t/a}{\log a + \psi'(t)}$$

with non-vanishing denominator.

Proof. Implicit differentiation of $\cos(t \log a + \psi(t)) = 0$ gives the formula. Non-vanishing of the denominator at points where $\cos(t \log a + \psi(t) = 0$ follows from the Maaß-Selberg relation, as follows. On one hand, for $\Im(s) > 0$ the higher Fourier terms of E_s do not vanish identically, so $\langle \wedge^a E_s, \wedge^a E_s \rangle > 0$. On the other hand, the Maaß-Selberg relation is

$$\begin{aligned}\langle \wedge^a E_s, \wedge^a E_r \rangle &= \frac{a^{s+\overline{r}-1}}{s+\overline{r}-1} + c_s \frac{a^{(1-s)+\overline{r}-1}}{(1-s)+\overline{r}-1} \\ &\quad + c_{\overline{r}} \frac{a^{s+(1-\overline{r})-1}}{s+(1-\overline{r})-1} + c_s c_{\overline{r}} \frac{a^{(1-s)+(1-\overline{r})-1}}{(1-s)+(1-\overline{r})-1}\end{aligned}$$

For $s = \frac{1}{2} + it + \varepsilon$ and $r = \frac{1}{2} + it$, this is

$$\langle \wedge^a E_s, \wedge^a E_r \rangle \;=\; \frac{a^\varepsilon}{\varepsilon} + c_s \frac{a^{-2it-\varepsilon}}{-2it-\varepsilon} + c_{\overline{r}} \frac{a^{2it+\varepsilon}}{2it+\varepsilon} + c_s c_{\overline{r}} \frac{a^{-\varepsilon}}{-\varepsilon}$$

As $\varepsilon \to 0$, the two middle terms go to

$$c_{\frac{1}{2}+it} \frac{a^{-2it}}{-2it} + c_{\frac{1}{2}-it} \frac{a^{2it}}{2it} = e^{-2i\psi(t)} \frac{a^{-2it}}{-2it} + e^{2\psi(t)} \frac{a^{2it}}{2it} = \frac{1}{t} \cdot \sin(2 \log a + 2\psi(t))$$

The vanishing condition is $\cos(t \log a + \psi(t)) = 0$, so $t \log a + \psi(t) \in \pi\mathbb{Z}$, so $2t \log a + 2\psi(t) \in 2\pi\mathbb{Z}$, and the sine function vanishes there. That is, the sum of these two terms is 0.

Modulo $O(\varepsilon^2)$, the sum of the first and last terms is

$$\begin{aligned}\frac{1}{\varepsilon}(a^\varepsilon - c_{\frac{1}{2}+it+\varepsilon} c_{\frac{1}{2}-it} a^{-\varepsilon}) \;&=\; \frac{1}{\varepsilon}\Big((1+\varepsilon \log a) - (c_{\frac{1}{2}+it} + \varepsilon c'_{\frac{1}{2}+it}) c_{\frac{1}{2}-it}(1-\varepsilon \log a)\Big) \\ &= \frac{1}{\varepsilon}\Big(1+\varepsilon \log a - (1+\varepsilon c'_{\frac{1}{2}+it} c_{\frac{1}{2}-it})(1-\varepsilon \log a)\Big) \;=\; 2\log a - \frac{c'_{\frac{1}{2}+it}}{c_{\frac{1}{2}+it}}\end{aligned}$$

since $c_s c_{1-s} = 1$. Since $c_{\frac{1}{2}+it} = e^{-2i\psi(t)}$ and c_s is holomorphic,

$$c'_{\frac{1}{2}+it} \;=\; \frac{d}{d(it)} c_{\frac{1}{2}+it} \;=\; -i\frac{d}{dt} c_{\frac{1}{2}+it} \;=\; -i\frac{d}{dt}\, e^{-2i\psi(t)} \;=\; -2\psi'(t)\cdot e^{-2i\psi(t)}$$

Thus, $c'_{\frac{1}{2}+it}/c_{\frac{1}{2}+it} = -2\psi'(t)$. That is, $\varepsilon \to 0$, these terms go to $2\log a + 2\psi'(t)$. That is, when $a^s + c_s a_s = 0$,

$$0 < \langle \wedge^a E_s, \wedge^a E_s\rangle = 2\log a + 2\psi'(t)$$

giving the non-vanishing of the denominator. *Q.E.D.*

Thus, to change a given zero t for cut-off height a by the expected gap amount $\pi/\log t$ at that height, change a by roughly

$$\frac{\text{gap}}{\text{derivative}} \sim \frac{\pi/(\log a + \log t)}{(-t/a)/(\log a + \log t)} \sim \frac{-\pi a}{t}$$

Corollary 66. *Fix $a_o > 1$. Given $\varepsilon > 0$ (and $\varepsilon < 1$), there is T_o sufficiently large such that, for every a in the range $1 < a \le a_o$, for every real zero $t = t(a)$ of $a^{\frac{1}{2}+it} + c_{\frac{1}{2}+it} a^{\frac{1}{2}-it}$,*

$$(1-\varepsilon)\cdot \frac{t/a}{\log t} \le -\frac{\partial t}{\partial a} \le (1+\varepsilon)\frac{t/a}{\log t}$$

Proof. Elementary from the above. *Q.E.D.*

Combining the previous with Corollary 60:

Theorem 67. *Given $\varepsilon > 0$, there is $T_o > 0$ sufficiently large such that, for two zeros $w_1 = \frac{1}{2}+i\tau_1$ and $w_2 = \frac{1}{2}+i\tau_2$ of θE_w with $T_o < \tau_1 < \tau_2$, if $\theta u_{\theta,w_1} \ge \theta u_{\theta,w_2}$, then we have the lower bound $|\tau_2 - \tau_1| \ge (1-\varepsilon)\pi/\log \tau_1$.*

Proof. Elementary from the above. *Q.E.D.*

As a very special case, relevant to the discrete spectrum (if any) of $\widetilde{S}_\theta$:

Corollary 68. *Given $\varepsilon > 0$, there is $T > 0$ sufficiently large such that, for two zeros $w_1 = \frac{1}{2} + i\tau_1$ and $w_2 = \frac{1}{2} + i\tau_2$ of θE_w with $T < \tau_1 < \tau_2$, if $\theta u_{\theta,w_1} = 0 = \theta u_{\theta,w_2}$, then we have the lower bound $|\tau_2 - \tau_1| \ge (1-\varepsilon)\pi/\log T$. That is, the discrete spectrum, if any, of $\widetilde{S}_\theta$, is λ_{w_j} with a lower bound $|w_{j+1} - w_j| \ge (1-\varepsilon)\pi/\log T$ for adjacent w_j and w_{j+1} at height T.*

Proof. This is a special case of the previous theorem and Corollary 52. *Q.E.D.*

7.5 Juxtaposition with pair-correlation

The asymptotic lower bound $\pi/\log T$ on spacing of consecutive spectral parameters w_j at height T for eigenvalues λ_{w_j} of $\widetilde{S}_\theta$ is in conflict with Montgomery's pair-correlation [21]. We continue to take θ to be a real-linear combination of Eisenstein-Heegner distributions, so that all previous results apply.

For example, assuming the Riemann Hypothesis and *pair correlation*:

Corollary 69. *At most 94% of the zeros of $\zeta(s)$ give eigenvalues $\lambda_s = s(1-s)$ for $\widetilde{S}_\theta$.*

Remark 70. *This shows that the optimistic simple version of the conjecture at the end of [5] cannot hold. Namely, with θ the Eisenstain-Dirac δ at $e^{2\pi i/3}$, it cannot be the case that all zeros ρ_j of $\zeta(s)$ give eigenvalues for $\widetilde{S}_\theta$.*

Remark 71. *Unless we believe that a subset of on-the-line zeros of $\zeta(s)$ is naturally distinguished, this might suggest that the discrete spectrum is empty.*

Proof. Let the imaginary parts of zeros be $\ldots \le \gamma_{-1} < 0 < \gamma_1 \le \gamma_2 \le \ldots$. Montgomery's pair correlation conjecture is that, for $0 \le \alpha < \beta$,

$$\#\Big\{m < n : 0 \le \gamma_m, \gamma_n \le T \text{ with } \frac{2\pi\alpha}{\log T} \le \gamma_n - \gamma_m \le \frac{2\pi\beta}{\log T}\Big\}$$
$$\sim \int_\alpha^\beta \Big(1 - (\frac{\sin \pi u}{\pi u})^2\Big)\, \mathrm{d}u$$

For example, the asymptotic fraction of pairs of zeros within half the average spacing $\frac{2\pi}{\log T}$ up to height T is

$$\int_0^{\frac{1}{2}} \Big(1 - (\frac{\sin \pi u}{\pi u})^2\Big)\, \mathrm{d}u \quad \approx \quad 0.11315 \quad > \quad 0$$

From the lower bound in the previous section, for at least one of every such pair (m, n) the corresponding zero cannot appear among discrete spectrum parameters w for $\widetilde{S}_\theta$. *Q.E.D.*

8 Spacing of zeros of $\zeta_k(s)$

Without assumptions on zeros $\zeta_k(s)$ as spectral parameters for any of the pseudo-Laplacians above, we have non-trivial corollaries on spacing of those zeros.

8.1 Exotic eigenfunction expansions and interleaving

Let θ continue to be a finite real-linear combination of Eisenstein-Heegner distributions. For this section, assume that the cut-off height a is above the highest Heegner point involved in θ, so that the determinant vanishing condition $\eta(v_{w,a})\theta(u_{\theta,w}) - \eta_a(u_{\theta,w})^2 = 0$ of (5.2), amplified as in the computations leading up to Corollary 28, simplifies to

$$a^{1-w}(a^w + c_w a^{1-w}) \times \Big(\frac{|\theta(1)|^2}{\langle 1,1\rangle \cdot \lambda_1 - \lambda_w)} + \frac{1}{4\pi i}\int_{(\frac{1}{2})} |\theta E_s|^2 \, \frac{\mathrm{d}s}{\lambda_s - \lambda_w}\Big)$$
$$= \frac{1}{2w-1}\,\big(a^{1-w}\theta E_w)\big)^2 \tag{8.1}$$

where

$$\theta E_s = \sum_d \nu_d \left(\frac{\sqrt{|d|}}{2}\right)^s \frac{\zeta(s)}{\zeta(2s)}\, L(s,\chi_d) \tag{8.2}$$

To apply the meromorphic continuation results of subsections 5.7 or 6.5, let

$$J_{\theta,w} \;=\; \frac{\theta(1)\cdot 1}{(\lambda_1 - \lambda_w)\cdot\langle 1,1\rangle} + \frac{1}{4\pi i}\int\limits_{(\frac{1}{2})} \frac{\theta E_{1-s}\cdot E_s - \theta E_{1-w}\cdot E_w}{\lambda_s - \lambda_w}\,\mathrm{d}s$$

This allows elementary algebraic rearrangement of the determinant-vanishing condition to a symmetrized form, as in subsection 5.7:

$$(a^{w-\frac{1}{2}} + c_w a^{\frac{1}{2}-w})\cdot J_{\theta,w} \;=\; (a^{w-\frac{1}{2}} - c_w a^{\frac{1}{2}-w})\cdot \frac{\theta E_{1-w}\cdot\theta E_w}{2(1-2w)}$$

On the critical line $w = \frac{1}{2} + i\tau$, letting $\psi(\tau) = \arg\xi(1+2i\tau)$ as above, this is

$$\cos\Big(\tau\log a + \psi(\tau)\Big)\cdot J_{\theta,w} \;=\; \sin\Big(\tau\log a + \psi(\tau)\Big)\cdot\frac{\theta E_{1-w}\cdot\theta E_w}{-4\tau}$$

Let $S_{\Theta,\theta}$ be the restriction of $-\Delta$ to test functions in the Lax-Phillips space $L^2_a(\Gamma\backslash\mathfrak{H})$ of pseudo-cuspforms which are also annihilated by θ. Let $\widetilde{S}_{\Theta,\theta}$ be its Friedrichs extension. We have a variant of earlier results:

Theorem 72. *The parameters $w = \frac{1}{2} + i\tau$ for eigenvalues $\lambda_w = w(1-w) < -1/4$ of $\widetilde{S}_{\Theta,\theta}$ are the solutions of*

$$\cos\Big(\tau \log a + \psi(\tau)\Big) \cdot J_{\theta,w} \;=\; \sin\Big(\tau \log a + \psi(\tau)\Big) \cdot \frac{\theta E_{1-w} \cdot \theta E_w}{-4\tau}$$

For such w, the eigenfunction is a linear combination $Au_{\theta,w} + Bv_{a,w}$ of the meromorphic continuations of $u_{\theta,w}$ and $v_{a,w}$, where A, B are not both 0, such that

$$\theta(Au_{\theta,w} + Bv_{a,w}) \;=\; 0 \;=\; \eta_a(Au_{\theta,w} + Bv_{a,w})$$

Proof. The equation $(\Delta - \lambda)u = 0$ has no solution $u \in \mathfrak{E}^2 \cap \mathfrak{E}^0_\Theta$ except $u = 0$, so the only possible λ_w-eigenvectors are solutions $u \in \mathfrak{E}^1$ to equations

$$(\Delta - \lambda_w)u \;=\; A \cdot \theta + B \cdot \eta_a$$

(with not both constants 0), with the additional condition $\theta u = 0 = \eta_a u$. In terms of spectral expansions of elements of $\mathfrak{E}^1$, as in subsection 4.1, if $\lambda_w < -1/4$ is an eigenvalue, then $(A\theta + B\eta_a)E_w = 0$. From the vector-valued meromorphic continuation results of subsection 6.5, the meromorphic continuation of

$$Au_{\theta,w} + Bv_{a,w} \;=\; \frac{A\theta + B\eta_a)(1) \cdot 1}{(\lambda_1 - \lambda_w) \cdot \langle 1, 1\rangle} + \frac{1}{4\pi i} \int_{(\frac{1}{2})} \frac{(A\theta + B\eta_a)E_{1-s} \cdot E_s}{\lambda_s - \lambda_w}\, ds$$

at w with $\Re(w) \le \frac{1}{2}$ is in $\mathfrak{E}^1$ when $(A\theta + B\eta_a)E_w = 0$. Further, $\eta_b(Au_{\theta,w} + Bv_{a,w}) = 0$ for all $b \ge a \gg_\theta 1$. Thus, when $\lambda_w < -1/4$ is an eigenvalue, some non-zero $Au_{\theta,w} + Bv_{a,w}$ is the eigenfunction. There exist such A, B, not both 0, exactly when the relevant two-by-two determinant vanishes, as above. That is, $\lambda_w < -1/4$ is an eigenvalue exactly when

$$\cos\Big(\tau \log a + \psi(\tau)\Big) \cdot J_{\theta,w} \;=\; \sin\Big(\tau \log a + \psi(\tau)\Big) \cdot \frac{\theta E_{1-w} \cdot \theta E_w}{-4\tau}$$

and the eigenfunction is a linear combination $Au_{\theta,w} + Bv_{a,w}$ of the meromorphic continuations. *Q.E.D.*

Via the Baire category theorem, given $a \ge 1$ and $\varepsilon > 0$, we can increase the cut-off height a by an amount less than ε so that $\theta E_{s_j} \ne 0$ for all parameters s_j with $\eta_a E_{s_j} = 0$, that is, for all eigenfunctions of the form $\wedge^a E_{s_j}$ for $\widetilde{S}_\Theta$. In the sequel, assume that such an adjustment has been made.

We have the *interleaving result:*

Theorem 73. *Between two consecutive zeros $\tau_j^{(a)}$ and $\tau_{j+1}^{(a)}$ of $\cos(\tau\log a+\psi(\tau))$ there is a unique τ such that $\lambda_{\frac{1}{2}+i\tau}$ is an eigenvalue of $\widetilde{S}_{\Theta,\theta}$.*

Proof. Since $\eta_a(Au_{\theta,w}+Bv_{a,w})=0$ implies that $f_w=Au_{\theta,w}+Bv_{a,w}$ is in $\mathfrak{E}^1\cap\mathfrak{E}^0_\Theta$, this f_w is expressible in terms of the orthonormal basis of eigenfunctions φ_j for $\widetilde{S}_\Theta$, in an expansion converging in $\mathfrak{E}^1$:

$$f_w \;=\; \sum_i \langle f_w,\varphi_i\rangle\cdot\varphi_i$$

Since we have arranged that $\theta E_{s_j}\neq 0$ for all s_j, necessarily $A\neq 0$, so $A=1$ without loss of generality. As earlier, let φ_i have eigenvalue $s_i(1-s_i)$, with $\Im(s_i)\le\Im(s_{i+1})$, and all φ_i real-valued. Let K be the kernel of η_a on $\mathfrak{E}^1\cap\mathfrak{E}^0_\Theta$, and $j:K\to\mathfrak{E}^1$ the inclusion, with adjoint $j^*:\mathfrak{E}^{-1}\to K^*$. Let $\widetilde{S}^\#_\Theta=:K\longrightarrow K^*$ be as in subsection 2.2. Then $(\widetilde{S}^\#_\Theta-\lambda_w)f_w=j^*(\theta+B\eta_a)$. Conveniently,

$$j^*(\theta+B\eta_a)\;=\;j^*\theta+B\cdot j^*\eta_a\;=\;j^*\theta+B\cdot 0\;=\;j^*\theta$$

The image $j^*(\theta+B\eta_a)=j^*\theta$ in K^* admits an eigenfunction expansion

$$j^*(\theta+B\eta_a)=j^*\theta\;=\;\sum_i(j^*\theta)(\varphi_i)\cdot\varphi_i\;=\;\sum_i\theta(j\varphi_i)\cdot\varphi_i$$

and then

$$\sum_i j^*(\theta+B\eta_a)(\varphi_i)\cdot\varphi_i\;=\;\sum_i(j^*\theta)(\varphi_i)\cdot\varphi_i\;=\;j^*\theta\;=\;(\widetilde{S}^\#_\Theta-\lambda_w)f_w$$

$$=\;(\widetilde{S}^\#_\Theta-\lambda_w)\sum_i\langle f_w,\varphi_i\rangle\cdot\varphi_i\;=\;\sum_i\langle f_w,\varphi_i\rangle\cdot(\widetilde{S}^\#_\Theta-\lambda_w)\varphi_i$$

$$=\;\sum_i\langle f_w,\varphi_i\rangle\cdot(\lambda_{s_i}-\lambda_w)\varphi_i$$

Thus, $\langle f_w,\varphi_i\rangle=(j^*\theta)\varphi_i/(\lambda_{s_i}-\lambda_w)$, and we have an expansion convergent in $\mathfrak{E}^1$:

$$f_w\;=\;\sum_i\frac{(j^*\theta)(\varphi_i)\cdot\varphi_i}{\lambda_{s_i}-\lambda_w}$$

The further condition $\theta(jf_w) = 0$ for f_w to be an eigenfunction for $\widetilde{S}_{\Theta,\theta}$ is

$$0 \;=\; \theta(jf_w) \;=\; (j^*\theta)f_w \;=\; \sum_i \frac{|(j^*\theta)\varphi_i|^2}{\lambda_{s_i} - \lambda_w}$$

By the intermediate value theorem, there is at least one w between any two s_i and s_{j+1}. The derivative of θf_w never vanishes, and no $(j^*\theta)(\varphi_i)$ is 0, so between s_i and s_{j+1} there is *exactly one* w on $\frac{1}{2} + i\mathbb{R}$ satisfying the equation. *Q.E.D.*

8.2 Dependence of eigenvalues on cut-off height

From the previous section, the necessary and sufficient condition on $w = \frac{1}{2} + i\tau$ for λ_w to be an eigenvalue of $\widetilde{S}_{a,\theta}$, equivalently, of $\widetilde{S}_{\geq a,\theta}$ is the vanishing condition

$$\cos\Big(\tau \log a + \psi(\tau)\Big) \cdot J(w) \;=\; \sin\Big(\tau \log a + \psi(\tau)\Big) \cdot \frac{\theta E_{1-w} \cdot \theta E_w}{2\tau}$$

A simultaneous zero of $J_{\theta,w}$ and θE_w makes λ_w an eigenvalue for $\widetilde{S}_\theta$ already, hence of $\widetilde{S}_{\Theta,\theta}$ and of $\widetilde{S}_{a,\theta}$, but we expect that there are few or no eigenvalues for $\widetilde{S}_\theta$. For λ_w *not* a simultaneous zero of $J_{\theta,w}$ and θE_w, with $w = \frac{1}{2} + i\tau$, this vanishing condition is equivalent to

$$\tan\Big(\tau \log a + \psi(\tau)\Big) \;=\; \frac{2\tau \cdot J(w)}{\theta E_w \cdot \theta E_{1-w}}$$

At least for $0 < \tau \in \mathbb{R}$, let

$$R(\tau \log \tau) \;=\; \frac{2\tau \cdot J(\frac{1}{2} + i\tau)}{\theta E_{\frac{1}{2}+i\tau} \cdot \theta E_{\frac{1}{2}-i\tau}}$$

denote the right-hand side.

Proposition 74. *For an eigenvalue $\lambda_{\frac{1}{2}+i\tau}$ of $\widetilde{S}_{\geq a,\theta}$ (equivalently, of $\widetilde{S}_{a,\theta}$),*

$$\frac{\partial \tau}{\partial a} \;=\; \frac{\frac{\tau}{a}(R(\tau \log \tau)^2 + 1)}{(\log \tau + 1) \cdot R'(\tau \log \tau) - (\log a + \psi'(\tau))(R(\tau \log \tau)^2 + 1)}$$

Corollary 75. *For all large* τ*,*

$$R'(\tau\log\tau) \;\le\; \frac{\log a + \psi'(\tau)}{\log\tau + 1}\cdot\big(R(\tau\log\tau)^2+1\big)$$

That is, for large τ*, the graph of* $t \to R(t\log t)$ *has slope essentially bounded above by the slope of* $t \to \tan(t\log t)$ *at the same height, since* $\psi'(\tau) \sim \log\tau$*.*

Proof. *(Of claim)* It makes sense to rescale the right-hand side of the eigenvalue condition, expressing it as a function of something asymptotic to $\tau\log\tau$, both because of the analogous scaling of the left-hand side, and because of the vertical asymptotics of zeros of θE_w. One simple choice is $R(\tau\log\tau)$, although one might use $R(\psi(\tau))$ instead. Using the simple rescaling, differentiating with respect to cut-off height a,

$$\Big(\frac{\tau}{a}+\frac{\partial\tau}{\partial a}(\log a+\psi'(\tau))\Big)\cdot\sec^2\Big(\tau\log a+\psi(\tau)\Big) \;=\; \frac{\partial\tau}{\partial a}\cdot(\log\tau+1)\cdot R'(\tau\log\tau)$$

so

$$\frac{\partial\tau}{\partial a}\Big((\log\tau{+}1)\cdot R'(\tau\log\tau)-(\log a{+}\psi'(\tau))\sec^2(\tau\log a{+}\psi(\tau))\Big) \;=\; \frac{\tau}{a}\sec^2(\tau\log a{+}\psi(\tau))$$

and

$$\frac{\partial\tau}{\partial a} \;=\; \frac{\frac{\tau}{a}\sec^2(\tau\log a+\psi(\tau))}{(\log\tau+1)\cdot R'(\tau\log\tau)-(\log a+\psi'(\tau))\sec^2(\tau\log a+\psi(\tau))}$$

Since $\sec^2\alpha = \tan^2\alpha+1$, by the relation $\tan(\tau\log a+\psi(\tau)) = R(\tau\log\tau)$, at such points $\sec^2(\tau\log a+\psi(\tau)) = R(\tau\log\tau)^2+1$. Thus,

$$\frac{\partial\tau}{\partial a} \;=\; \frac{\frac{\tau}{a}(R(\tau\log\tau)^2+1)}{(\log\tau+1)\cdot R'(\tau\log\tau)-(\log a+\psi'(\tau))(R(\tau\log\tau)^2+1)}$$

which is the assertion of the claim. *Q.E.D.*

Proof. *(Of corollary)* From the min-max principle, when $\partial\tau/\partial a$ it exists, it satisfies $\partial\tau/\partial a \le 0$. We also have the *interleaving* of parameters τ for eigenvalues $\lambda_{\frac{1}{2}+i\tau}$ of $\widetilde{S}_{a,\theta}$ between those of $\widetilde{S}_\Theta$. The parameter values τ for $\widetilde{S}_\Theta$ can be adjusted by changing cut-off height a: changing a by about $\pi a/\tau$ moves the zeros by about the average gap amount $\pi/\log\tau$. Thus, given large τ, by the intermediate value theorem, we need increase a only slightly to make $\lambda_{\frac{1}{2}+i\tau}$ an eigenvalue for $\widetilde{S}_{a,\theta}$. Thus,

$$R'(\tau\log\tau) \;\le\; \frac{\log a + \psi'(\tau)}{\log\tau + 1}\cdot\left(R(\tau\log\tau)^2 + 1\right)$$

holds for *all* large τ. *Q.E.D.*

8.3 Sample unconditional results on spacing of zeros

Without any assumption that zeros of θE_w do or do not give eigenvalues λ_w for a self-adjoint operator, we have illustrative corollaries about spacing of zeros of θE_w.

Corollary 76. *Let $t < t'$ be large such that $\frac{1}{2}+it$ and $\frac{1}{2}+it'$ are adjacent on-line zeros of θE_w, and neither a zero of $J_{\theta,w}$. Suppose there is a unique zero $\frac{1}{2}+i\tau_o$ of $J_{\theta,\frac{1}{2}+i\tau}$ between $\frac{1}{2}+it$ and $\frac{1}{2}+it'$, and $\frac{\partial}{\partial\tau}J_{\theta,\frac{1}{2}+i\tau} > 0$. Then $|t'-t| \ge \frac{\pi}{\log t}\cdot(1+O(\frac{1}{\log\log t}))$. That is, in this configuration, the distance between consecutive zeros must be at least the average.*

Corollary 77. *Let $t < t'$ be large such that $\frac{1}{2}+it$ and $\frac{1}{2}+it'$ are adjacent on-line zeros of θE_w, and neither a zero of $J_{\theta,w}$. Suppose there is no zero of $J_{\theta,w}$ on the critical line between $\frac{1}{2}+it$ and $\frac{1}{2}+it'$, but there is a pair of off-line zeros $\frac{1}{2}\pm\varepsilon+i\tau_o$ of $J_{\theta,w}$ with $t < \tau_o < t'$ and ε very small. Then $|t'-t| \ge \frac{\pi}{2\log t}\cdot(1+O(\frac{1}{\log\log t}))$. That is, in this configuration, the distance between the consecutive zeros must be least essentially half the average.*

Proof. (Of both) In the scenario of the first corollary, this bound implies that $R(t\log t)$ cannot get from $-\infty$ (at $\frac{1}{2}+i\tau$) to $+\infty$ (at $\frac{1}{2}+i\tau'$) much faster than $\tan(t\log t)$, which is by a change of $\pi/\log t$.

In the scenario of the second corollary, $R(t\log t)$ goes to $\pm\infty$ (with the same sign) approaching t from the right and t' from the left, and is very near 0 at τ_o. Depending on the sign, the bound by comparison to the *tangent* function implies that either from t to τ_o, or else from τ_o to t', the function $R(t)$ cannot change faster than $\tan(t\log t)$. *Q.E.D.*

Remark 78. *Similar, somewhat more complicated, and perhaps less interesting, corollaries hold, as well, for off-the-line zeros.*

References

[1] R. BACKLUND, Über die Nullstellen der Riemannschen Zetafunktion, *Acta Math.* **41** (1918), 345–75.

[2] N. BOURBAKI, *Topological Vector Spaces, ch. 1–5*, Springer-Verlag, 1987.

[3] F. CARLSON, Über die Nullstellen der DIRICHLETschen Reihen und der RIEMANNschen ζ-funktion, *Arkiv för Mat. Astr. och Fysik*, **15** (1920), No. 20.

[4] Y. COLIN DE VERDIÈRE, Une nouvelle démonstration du prolongement méromorphe des séries d'Eisenstein, *C. R. Acad. Sci. Paris Sér. I Math.* **293** (1981), no. 7, 361–363.

[5] Y. COLIN DE VERDIÈRE, Pseudo-laplaciens, I, II, *Ann. Inst. Fourier (Grenoble)*, **32** (1982) no. 3, 275–286; *ibid* **33** (1983) no. 2, 87–113.

[6] L. FADDEEV, Expansion in eigenfunctions of the Laplace operator on the fundamental domain of a discrete group on the Lobacevskii plane, *AMS Transl. Trudy* (1967), 357-386.

[7] J. D. FAY, Fourier coefficients of the resolvent for a Fuchsian group, *J. für reine und angewandte Math. (Crelle)* **293-4** (1977), 143–203.

[8] L. FOX, P. HENRICI, C. MOLER, Approximations and bounds for eigenvalues of elliptic operators, *SIAM J. Numerical Analysis* **4** (1967), 89–102.

[9] K. FRIEDRICHS, Spektraltheorie halbbeschrañkter Operatoren, I, II, *Math. Ann.* **109** (1934), 465–487; *ibid* **110** (1935), 777–779.

[10] P. GARRETT, Modern Analysis of Automorphic Forms, by Example, two volumes, Cambridge University Press, 2018.

[11] I. M. GELFAND, Sur un lemme de la theorie des espaces lineaires, *Comm. Inst. Sci. Math. de Kharkkoff*, no. 4, **13** (1936), 35–40.

[12] D. GOLDFELD and J. HOFFSTEIN, Eisenstein series of $\frac{1}{2}$-integral weight and the mean value of real Dirichlet L-series, *Inventiones math.* **80** (1985), 185–208.

[13] A. GROTHENDIECK, Sur certains espaces de fonctions holomorphes, I, II, *J. Reine Angew. Math.* **192** (1953), 35–64 and 77–95.

[14] H. HAAS, Numerische Berechnung der Eigenwerte der Differentialgleichung $y^2\Delta u + \lambda u = 0$ für ein unendliches Gebiet im $\mathbb{R}^2$, Diplomarbeit, Universität Heidelberg (1977), 155 pp.

[15] G. H. HARDY and J. E. LITTLEWOOD, Contributions to the theory of the Riemann zeta-function and the theory of the distribution of primes, *Acta Math.* **41** (1918), 119–196.

[16] D. HEJHAL, Some observations concerning eigenvalues of the Laplacian and Dirichlet L-series, in *Recent Progress in Analytic Number Theory*, ed. H. Halberstam and C. Hooley, vol. 2, Academic Press, NY, 1981, 95–110.

[17] D. HEJHAL D. Hejhal, On a result of G. Pólya concerning the Riemann ξ-function, *J. An. Math.* **55** (1990), 59–95.

[18] H. IWANIEC, *Introduction to the Spectral Theory of Automorphic Functions*, Graduate Studies in Mathematics, Second edition **53**, American Mathematical Society, Providence, RI; Revista Matemática Iberoamericana, Madrid, 2002. xii+220 pp.

[19] T. KATO, *Perturbation theory for linear operators*, Springer-Verlag, 1966, second edition, 1976, reprinted 1995.

[20] P. LAX, R. PHILLIPS, *Scattering Theory for Automorphic Functions*, Ann. of Math. Studies 87, Princeton, NJ, 1976.

[21] H.L. MONTGOMERY, The pair correlation of zeros of the zeta-function, *Proc. Symp. Pure Math.* **24**, AMS, Providence R.I., 1973, 181–193.

[22] H. NEUNHÖFFER, Über die analytische Fortsetzung von Poincaréreihen, *Sitzungsberichte Heidelberg Akad. Wiss.* (1973), no. 2, 33–90.

[23] D. NIEBUR, A class of nonanalytic automorphic functions, *Nagoya Math. J.* **52** (1973), 133–145.

[24] A.M. ODLYZKO, [Scanned images and transcripts of letters to and from G. Polya 1982-3, and O. Tausky-Todd 1982, attempting to trace the origins of the so-called *Hilbert-Polya conjecture*.] http://www.dtc.umn.edu/ odlyzko/polya/

[25] B. J. PETTIS, On integration in vector spaces, *Trans. A.M.S.* **44** (1936), 277–304.

[26] W. RUDIN, *Functional Analysis*, second edition, McGraw-Hill, 1991.

[27] E. C. TITCHMARSH (edited and with a preface by D. R. HEATH-BROWN, *The theory of the Riemann zeta- function*, 2nd ed.n The Clarendon Press. Oxford University Press, New York, 1986. First edition, 1951, successor to *The Zeta-Function of Riemann*, 1930.

[28] A. I. VINOGRADOV and TAKHTADZHYAN, On analogues of the Vinogradov–Gauss formula in the critical strip, *Tr. Mat. Inst. Steklov, Akad. Nauk SSSR* **158** (1981). English Translation in Proceedings of the Steklov Institute of Mathematics, (1983), Issue 4, 47–71. ©1984 American Mathematical Society 0081-5438.