Let \( \mathfrak{o} \) be the ring of algebraic integers in a number field \( k \), \( \mathbf{O} \) the ring of integers in a finite Galois extension \( K \) of \( k \), with Galois group \( G \). For a prime \( p \) in \( \mathbf{O} \) lying over a prime \( p \) in \( \mathfrak{o} \), the decomposition (sub-) group \( G_P \subset G \) is the subgroup stabilizing (not necessarily pointwise fixing) \( P \).

The fixed field \( L = K^{G_P} \) of \( G_P \) has the property that it is the largest subfield of \( K \) (containing \( k \)) such that \( P \) is the only prime of \( \mathbf{O} \) lying over \( Q = P \cap L \). The residue fields are related by \( \mathfrak{o}/p = \mathbf{O}'/Q \), where \( \mathbf{O}' \) is the ring of algebraic integers in \( L \).

Then \( G_P \) acts on the residue field \( \mathfrak{o}/P \), and in fact surjects to the Galois group of \( \mathfrak{o}/P \) over \( \mathfrak{o}/p \). The kernel \( I_P \) is called the inertia subgroup, which is trivial if \( P \) is unramified over \( p \), so the inertia subgroup is trivial for almost all \( p \).

Let \( \mathfrak{o}/p \) have \( q \) elements. Then the Galois group of \( \mathfrak{o}/P \) over \( \mathfrak{o}/p \) is generated by the Frobenius automorphism \( \alpha \to \alpha^q \). Let \( \Phi_P \) be the inverse image of \( \alpha \to \alpha^q \) in the decomposition group \( G_P \). There are other notations as well, such as \( \Phi_P = (P, K/k) \).

For \( P \) ramified over \( p \), we only have an \( I_P \)-coset rather than an element, and more complicated considerations are necessary. We won’t worry about this, since at worst only finitely many primes are ramified.

Since the Galois group of \( K/k \) is transitive on primes \( P \) lying over \( p \), all the Frobenius elements \( \Phi_P \) for \( P \) over \( p \) are conjugate. Thus, attached to the prime \( p \) downstairs is a conjugacy class of Frobenius elements in \( \text{Gal}(K/k) \).

When the Galois group is abelian, the conjugacy class of Frobenius elements \( \Phi_P \) for primes \( P \) over \( p \) necessarily consists of a single element, called the Artin symbol.

We will associate to a finite-dimensional representation \( \rho \) of \( \text{Gal}(K/k) \) a Dirichlet series with Euler product, the Artin L-function, as follows. To conform with standard usage, now use \( v \) to denote a (finite) place of \( \mathfrak{o}, p_v \) the associated prime ideal in \( \mathfrak{o}, q_v \) the residue field cardinality \( \mathfrak{o}/p_v \), and \( \Phi_v \) the conjugacy class of Frobenius elements attached to \( p_v \), for \( v \) unramified in the extension \( K/k \). Let \( S \) be the finite set of (finite) places ramified in \( K/k \). Define the Artin L-function

\[
L(s, \rho) = \prod_{v \in S} \frac{1}{\det(1 - q_v^{-s} \Phi_v)}
\]

The indicated determinant is indeed well-defined since does only depend upon the conjugacy class.

Artin conjectured in the 1930’s that for \( \rho \) irreducible and not the trivial representation the L-function is entire.

For abelian \( \text{Gal}(K/k) \) classfield theory proves that these L-functions are among the L-functions attached to Hecke characters, and Hecke (and Iwasawa and Tate) proved that Hecke L-functions have analytic continuations, proving Artin’s conjecture in this case. In the abelian Galois group case Artin L-functions are called abelian L-functions.

For non-abelian \( \text{Gal}(K/k) \), R. Brauer proved that there is a meromorphic continuation by showing that these L-functions are quotients of products of abelian L-functions attached to intermediate fields, by proving that all irreducibles \( \rho \) of the Galois group can be expressed as \( \mathbb{Z} \)-linear combinations of induced representations of one-dimensional representations on subgroups. This does not prove the entire-ness, however.

In the 1960’s R. Langlands offered a new viewpoint on Artin’s conjecture, namely that for an \( n \)-dimensional irreducible \( \rho \) the Artin L-function should be equal to an L-function associated to a cuspform (or cuspidal automorphic representation) on \( GL(n) \), whose analytic continuation had been proven just about then, by Jacquet-Piatetski-Shapiro-Shalika, and also by Jacquet-Godement. That is, Langlands changed the issue to assertion that an L-function coming from Galois theory (the Artin L-function) should be equal to an analytically defined L-function (the automorphic one).

Except for the abelian case and two-dimensional examples, very little has been proven.