(*July 9, 2010*)

Basic Rankin-Selberg

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We present the simplest possible example of the Rankin-Selberg method, namely for a pair of holomorphic modular forms for $SL(2,\mathbb{Z})$, treated independently in 1939 by Rankin and 1940 by Selberg. (Rankin has remarked that the general idea came from his advisor and mentor, Ingham.) We also recall a proof of the analytic continuation of the relevant Eisenstein series. That is, we consider the simplest instance of an identity

$$\langle f \cdot E_s, g\rangle = \text{ L-function}$$

where f, g are cuspforms and E_s is an Eisenstein series. Or, contrariwise, one might consider

$$\langle \text{Res}^G_H E_s, f\rangle = \text{ L-function}$$

where the notation indicates that the Eisenstein series on a larger group was restricted to a smaller group H and there integrated against a cuspform f. One should note that it is not at all obvious that such integrals should yield L-functions or Dirichlet series of any sort.

Let f, g be two holomorphic cuspforms on the upper half-plane $\mathfrak{H}$ of weight $2k$ for $SL(2,\mathbb{Z})$, with Fourier expansions

$$f(z) = \sum_{n>0} a_n\, e^{2\pi i n z}$$

$$g(z) = \sum_{n>0} b_n\, e^{2\pi i n z}$$

Let $\Gamma = SL(2,\mathbb{Z})$ for brevity, and let

$$P = \{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \in \Gamma\}$$

Then define an Eisenstein series E_s by

$$E_s(z) = \sum_{\gamma\in P\backslash\Gamma} \text{Im}\,(\gamma z)^s$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ acts on z in the upper half-plane as usual by linear fractional transformations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(z) = \frac{az+b}{cz+d}$$

Noting that

$$\text{Im}\begin{pmatrix} a & b \\ c & d \end{pmatrix}(z) = \frac{\text{Im}\,(z)}{|cz+d|^2}$$

one can verify that the series defining E_s converges absolutely (and uniformly on compacta) for $\text{Re}\,(s) > 1$. In that region it is straightforward to verify that E_s is $SL(2,\mathbb{Z})$-invariant.

Further, it is standard (and proven below as an appendix) that E_s has an analytic continuation to $s \in \mathbb{C}$ with a functional equation described as follows. First, let

$$\xi(s) = \pi^{-s/2}\,\Gamma(\frac{s}{2})\,\zeta(s)$$

be the usual zeta function

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s}$$

along with its gamma factor $\pi^{-s/2}\Gamma(\frac{s}{2})$. Recall the functional equation of zeta

$$\xi(1-s) = \xi(s)$$

The functional equation of the Eisenstein series is

$$\xi(2s)\, E_s = \xi(2-2s)\, E_{1-s}$$

and the expression $\xi(2s)E_s$ has poles only at $s = 0, 1$. By the identity principle of complex analysis the analytic continuation is also $SL(2, \mathbb{Z})$-invariant.

It is also important to verify that both in the convergent region and when analytically continued the Eisenstein series is of *moderate growth* as $\mathrm{Im}\,(x)$ goes to $+\infty$, meaning that

$$|E_s(z)| = O(y^N)$$

for some N, as $y \to +\infty$, for $z = x + iy$ in the standard fundamental domain

$$F = \{z \in \mathfrak{H} : |x| \leq \frac{1}{2}, |z| \geq 1\}$$

for $SL(2, \mathbb{Z})\backslash\mathfrak{H}$.

The Petersson inner product for weight $2k$ modular forms is

$$\langle f, g \rangle = \int_{\Gamma\backslash\mathfrak{H}} f(z)\, \overline{g(z)}\, y^{2k}\, \frac{dx\, dy}{y^2}$$

where $dx\, dy/y^2$ is a $SL(2, \mathbb{R})$-invariant measure on $\mathfrak{H}$. Note that the expression

$$f(z)\, \overline{g(z)}\, y^{2k}$$

is Γ-invariant.

The simplest *Rankin-Selberg integral* (from Rankin 1939 and Selberg 1940) is

$$\langle f \cdot E_s, g \rangle$$

This integral converges for all $s \in \mathbb{C}$ since on the fundamental domain F the analytically continued Eisenstein series is of moderate growth, and the cuspforms are of rapid decay in the sense that on F

$$|f(z)| = O(y^{-N})$$

for all N.

[0.0.1] Theorem:

$$\langle f \cdot E_s, g \rangle = (4\pi)^{-(s+2k-1)}\, \Gamma(s + 2k - 1) \sum_{n \geq 1} \frac{a_n\, \overline{b_n}}{n^{s+2k-1}}$$

The function

$$\xi(2s)\, \langle f \cdot E_s, g \rangle = (4\pi)^{-(s+2k-1)}\, \xi(2s)\, \Gamma(s + 2k - 1) \sum_{n \geq 1} \frac{a_n\, \overline{b_n}}{n^{s+2k-1}}$$

has an analytic continuation to $\mathbb{C}$ with poles at most at $s = 0, 1$.

Proof: For an integrable P-invariant function φ on $\mathfrak{H}$ we have a general identity akin to Fubini's theorem

$$\int_{P\backslash\mathfrak{H}} \varphi(z)\, \frac{dx\, dy}{y^2} = \int_{\Gamma\backslash\mathfrak{H}} \sum_{\gamma\in P\backslash\Gamma} \varphi(\gamma z)\, \frac{dx\, dy}{y^2}$$

Applying this to

$$\varphi(z) = y^s \cdot f(z)\overline{g(z)}\, y^{2k}$$

we obtain

$$\int_{P\backslash\mathfrak{H}} y^s\, f(z)\overline{g(z)}\, y^{2k}\, \frac{dx\, dy}{y^2} \;=\; \langle f\cdot E_s, g\rangle$$

For fundamental domain for $P\backslash\mathfrak{H}$ we take

$$\Phi = \{z = x + iy \in \mathfrak{H} : 0 \le x \le 1\}$$

This region is a cartesian product, and we may integrate first in x, using the Fourier expansions of f and g

$$\int_{P\backslash\mathfrak{H}} y^s\, f(z)\overline{g(z)}\, y^k\, \frac{dx\, dy}{y^2} = \sum_{m,n\ge 1} a_m\overline{b_n} \int_{y>0} y^{s+2k-1} e^{-2\pi(m+n)y} \left(\int_{0\le x\le 1} e^{2\pi i(m-n)x}\, dx\right) \frac{dy}{y}$$

$$= \sum_{n\ge 1} a_n\overline{b_n} \int_{y>0} y^{s+2k-1} e^{-4\pi n y}\, \frac{dy}{y} = (4\pi)^{-(s+2k-1)} \sum_{n\ge 1} a_n\overline{b_n}\, n^{-(s+2k-1)} \int_{y>0} y^{s+2k-1} e^{-4\pi y}\, \frac{dy}{y}$$

$$= (4\pi)^{-(s+2k-1)}\, \Gamma(s + 2k - 1) \sum_{n\ge 1} \frac{a_n\overline{b_n}}{n^{s+2k-1}}$$

This holds by direct computation for $\mathrm{Re}(s) > 1$, and then by the identity principle for the analytically continued Eisenstein series. The integral is absolutely convergent for all $s \in C$ away from the poles of the Eisenstein series, since the Eisenstein series is of moderate growth and the cuspforms are of rapid decay. ///

Recall that for $f(z) = \sum_n a_n e^{2\pi i n z}$ a *normalized Hecke eigenfunction* we have an *Euler product* expansion of the associated L-function

$$L_f(s) = \sum_n \frac{a_n}{n^s} = \prod_{\text{prime } p} \frac{1}{1 - a_p p^{-s} + p^{2k-1-2s}}$$

The quadratic denominator attached to the prime p factors

$$\frac{1}{1 - a_p p^{-s} + p^{2k-1-2s}} = \frac{1}{(1-\alpha_p p^{-s})(1-\beta_p p^{-s})}$$

The Dirichlet series of the theorem admits an Euler product:

[0.0.2] Corollary: For f and g normalized Hecke eigenfunctions,

$$\xi(2s - 4k + 2)\, \langle f\cdot E_{s-2k+1}, g\rangle$$

has the Euler product expansion

$$(2\pi)^{-s}\, \Gamma(s) \prod_{\text{prime } p} \frac{1}{(1-\alpha\gamma p^{-s})(1-\alpha\delta p^{-s})(1-\beta\gamma p^{-s})(1-\beta\delta p^{-s})}$$

where α, β are attached to f and γ, δ are attached to g, and where we have suppressed the index p on $\alpha, \beta, \gamma, \delta$.

Proof: We start with the computation of the theorem, and factor the Dirichlet series

$$\sum_{n \geq 1} \frac{a_n \, \overline{b_n}}{n^s}$$

over primes. The Hecke eigenfunction assumptions that

$$L_f(s) = \sum_n \frac{a_n}{n^s} = \prod_{\text{prime } p} \frac{1}{(1 - \alpha_p p^{-s})(1 - \beta_p p^{-s})}$$

$$L_g(s) = \sum_n \frac{b_n}{n^s} = \prod_{\text{prime } p} \frac{1}{(1 - \gamma_p p^{-s})(1 - \delta_p p^{-s})}$$

yield (by expanding the geometric series)

$$a_{p^\ell} = \frac{\alpha_p^{\ell+1} - \beta_p^{\ell+1}}{\alpha_p - \beta_p} \qquad b_{p^\ell} = \frac{\gamma_p^{\ell+1} - \delta_p^{\ell+1}}{\gamma_p - \delta_p}$$

Note that the assumption that $g(z)$ is a normalized Hecke eigenfunction implies that its Fourier coefficients are *real*, so we can drop the complex conjugation. The weak multiplicativity of the Fourier coefficients immediately gives

$$\sum_{n \geq 1} \frac{a_n \, b_n}{n^s} = \prod_{\text{prime } p} \sum_{\ell \geq 0} \frac{a_{p^\ell} \, b_{p^\ell}}{(p^\ell)^s} = \prod_{\text{prime } p} \sum_{\ell \geq 0} (p^\ell)^{-s} \frac{\alpha_p^{\ell+1} - \beta_p^{\ell+1}}{\alpha_p - \beta_p} \cdot \frac{\gamma_p^{\ell+1} - \delta_p^{\ell+1}}{\gamma_p - \delta_p}$$

For fixed prime p, the inner sum over ℓ is essentially four geometric series. Let $X = p^{-s}$ and suppress the subscript p. Sum these geometric series separately

$$\sum_{\ell \geq 0} X^\ell \cdot \frac{\alpha^{\ell+1} - \beta^{\ell+1}}{\alpha - \beta} \cdot \frac{\gamma^{\ell+1} - \delta^{\ell+1}}{\gamma - \delta} = \frac{1}{\alpha - \beta} \cdot \frac{1}{\gamma - \delta} \cdot \left(\frac{\alpha\gamma}{1 - \alpha\gamma X} - \frac{\alpha\delta}{1 - \alpha\delta X} - \frac{\beta\gamma}{1 - \beta\gamma X} + \frac{\beta\delta}{1 - \beta\delta X} \right)$$

The algebraic identity that makes this whole computation work is

$$\frac{1}{\gamma - \delta} \cdot \left(\frac{\alpha\gamma}{1 - \alpha\gamma X} - \frac{\alpha\delta}{1 - \alpha\delta X} \right) = \frac{1}{\gamma - \delta} \cdot \frac{\alpha\gamma - \alpha^2\gamma\delta X - \alpha\delta + \alpha^2\gamma\delta X}{(1 - \alpha\gamma X)(1 - \alpha\delta X)} = \frac{\alpha}{(1 - \alpha\gamma X)(1 - \alpha\delta X)}$$

Thus, the right-hand side of the expression obtained from the sum over ℓ for fixed prime p is

$$\frac{1}{\alpha - \beta} \cdot \left(\frac{\alpha}{(1 - \alpha\gamma X)(1 - \alpha\delta X)} - \frac{\beta}{(1 - \beta\gamma X)(1 - \beta\delta X)} \right)$$

Putting everything over a common denominator, this is

$$\frac{1}{\alpha - \beta} \cdot \frac{(\alpha - (\beta\gamma + \beta\delta)\alpha X + \alpha\beta^2\gamma\delta X^2) - (\beta - (\alpha\gamma + \alpha\delta)\beta X + \beta\alpha^2\gamma\delta X^2)}{(1 - \alpha\gamma X)(1 - \alpha\delta X)(1 - \beta\gamma X)(1 - \beta\delta X)}$$

The middle terms cancel, leaving

$$\frac{1}{\alpha - \beta} \cdot \frac{(\alpha + \alpha\beta^2\gamma\delta X^2) - (\beta + \beta\alpha^2\gamma\delta X^2)}{(1 - \alpha\gamma X)(1 - \alpha\delta X)(1 - \beta\gamma X)(1 - \beta\delta X)} = \frac{1 - \alpha\beta\gamma\delta X^2}{(1 - \alpha\gamma X)(1 - \alpha\delta X)(1 - \beta\gamma X)(1 - \beta\delta X)}$$

Using $\alpha\beta = p^{2k-1}$ and $\gamma\delta = p^{2k-1}$ this is

$$\frac{1-p^{4k-2}X^2}{(1-\alpha\gamma X)(1-\alpha\delta X)(1-\beta\gamma X)(1-\beta\delta X)} = \frac{1-p^{4k-2-2s}}{(1-\alpha\gamma p^{-s})(1-\alpha\delta p^{-s})(1-\beta\gamma p^{-s})(1-\beta\delta p^{-s})}$$

The numerator is exactly the p-factor of $\zeta(2s+2-4k)$, as indicated in the assertion of the theorem, and the gamma factor (with power of π) is as asserted. ///

[0.0.3] Remark: The *tensor product L-function*

$$L(f\otimes g, s) = \prod_{\text{prime } p} \frac{1}{\det(1_4 - p^{-s}\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix}\otimes\begin{pmatrix}\gamma & 0\\ 0 & \delta\end{pmatrix})}$$

$$= \prod_{\text{prime } p}\frac{1}{(1-\alpha\gamma p^{-s})(1-\alpha\delta p^{-s})(1-\beta\gamma p^{-s})(1-\beta\delta p^{-s})}$$

attached to f and g was only recently shown by Ramakrishnan to be attached to an automorphic form on $GL(4)$, matching general (mostly unproven) conjectures of Langlands.

[0.0.4] Remark: Rankin's original purpose in considering the tensor product L-function was to approach Ramanujan's conjecture on the size of Hecke eigenvalues, and Rankin did achieve the best result at that time. More recent work of Shahidi and Kim-Shahidi is a (modernized) continuation of this theme, though by now the L-functions themselves have acquired their own interest.

Now we sketch a method for proving analytic continuation and functional equation of the Eisenstein series, taking the simplest possible case, following Godement's 1966 rewriting of an idea that occurred in Rankin's 1939 paper, if not earlier.

Again, let P be the upper triangular matrices in $SL(2,\mathbb{Z})$. The Eisenstein series may be expressed as

$$E_s(z) = \sum_{\gamma\in P\backslash SL(2,\mathbb{Z})} \mathrm{Im}\,(\gamma(z))^s = \frac{1}{2}\cdot\sum_{c,d\text{ coprime}}\frac{y^s}{|cz+d|^{2s}}$$

where $\gamma = \begin{pmatrix}a & b\\ c & d\end{pmatrix}$. The factor of $\frac{1}{2}$ is present because $(-c,-d)$ gives a contribution identical to that of (c,d). A standard and convenient related form is

$$\tilde{E}_s(z) = 2\,\zeta(2s)\,E_s(z) = \sum_{(c,d)\neq(0,0)}\frac{y^s}{|cz+d|^{2s}}$$

where (c,d) is summed over *all* non-zero vectors in $\mathbb{Z}^2$, dropping the coprimality condition, and dropping the factor of $\frac{1}{2}$. Both expressions converge for $\mathrm{Re}\,(s) > 1$.

[0.0.5] Theorem: The function

$$\pi^{-s}\Gamma(s)\tilde{E}_s(z) = 2\,\pi^{-s}\Gamma(s)\zeta(2s)\,E_s(z)$$

has a meromorphic continuation to $s\in\mathbb{C}$ with poles only at $s = 0, 1$, which are simple, and is invariant under

$$s \to 1-s$$

The residue of $E_s(z)$ at $s=1$ is the constant function $3/\pi$. The function $E_s(z)$ has no pole in $\mathrm{Re}\,(s) > \frac{1}{2}$ other than at $s=1$.

Proof: We prove the meromorphic continuation and functional equation. For $(c,d) = v \in \mathbb{R}^2$, consider the Gaussian

$$\varphi(v) = e^{-\pi|v|^2} = e^{-\pi(c^2+d^2)}$$

where $v \to |v|$ is the usual length function on $\mathbb{R}^2$. For $g \in GL(2,\mathbb{R})$, define

$$\Theta(g) = \sum_{v\in\mathbb{Z}^2} \varphi(v\cdot g) = \sum_{(c,d)\in\mathbb{Z}^2} e^{-\pi|(c,d)g|^2}$$

where we view $v \in \mathbb{R}^2$ as being a row vector. Consider the *integral representation* (a Mellin transform)

$$\int_0^\infty t^{2s}\,(\Theta(tg)-1)\,\frac{dt}{t}$$

where the t in the argument of Θ simply acts by scalar multiplication on $g \in GL(2,\mathbb{R})$. On one hand, integrating term-by-term gives

$$\int_0^\infty t^{2s}\,(\Theta(tg)-1)\,\frac{dt}{t} = \sum_{v\neq(0,0)} \int_0^\infty t^{2s}\,e^{-\pi|tvg|^2}\,\frac{dt}{t}$$

Since

$$\pi|tvg|^2 = (t\cdot\sqrt{\pi}|vg|)^2$$

we can change variables by replacing t by $t/(\sqrt{\pi}|vg|)$ to obtain

$$\sum_{v\neq(0,0)} (\sqrt{\pi}|vg|)^{-2s} \int_0^\infty t^{2s}\,e^{-t^2}\,\frac{dt}{t} = \frac{1}{2}\pi^{-s} \sum_{v\neq(0,0)} |vg|^{-2s} \int_0^\infty t^s\,e^t\,\frac{dt}{t}$$

$$= \frac{1}{2}\pi^{-s}\Gamma(s) \sum_{v\neq(0,0)} |vg|^{-2s}$$

Now we want $g \in SL(2,\mathbb{R})$ of a simple sort and chosen to map i in the upper half-plane to $x+iy$ (acting by linear fractional transformations). One reasonable choice is

$$g = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{y} & 0 \\ 0 & 1/\sqrt{y} \end{pmatrix}$$

Using this choice of G and writing out $v = (c,d)$ gives

$$vg = (c,d)g = (\,c \quad d\,) \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{y} & 0 \\ 0 & 1/\sqrt{y} \end{pmatrix} = (c\sqrt{y},\ (cx+d)/\sqrt{y})$$

and thus

$$\sum_v |vg|^{-2s} = \sum_v |(c\sqrt{y}, (cx+d)/\sqrt{y})|^{-2s} = \sum_v (c^2y + (cx+d)^2/y)^{-s}$$

$$= \sum_v \frac{y^s}{(c^2y^2+(cx+d)^2)^s} = \sum_v \frac{y^s}{|ciy+cx+d|^{2s}} = \sum_v \frac{y^s}{|cz+d|^{2s}}$$

Thus, we see that the integral representation yields the Eisenstein series $\tilde{E}$ with a leading power of π and a gamma function.

On the other hand, to prove the meromorphic continuation, we use the integral representation of the Eisenstein series in terms of Θ. We essentially follow an argument of Riemann for the Euler-Riemann zeta function, first breaking the integral into two parts, one from 0 to 1, and the other from 1 to $+\infty$. Keep $g \in SL(2,\mathbb{R})$ in a compact subset of $SL(2,\mathbb{R})$. Then

$$\int_1^\infty t^{2s}\,(\Theta(tg) - 1)\,\frac{dt}{t} = \text{ entire in } s$$

since elementary estimates show that the integral is uniformly and absolutely convergent. Indeed, this integral is a smooth function of g for all $s \in \mathbb{C}$. Apply Poisson summation to the kernel: first note that the Gaussian $\varphi(v) = e^{-\pi|v|^2}$ is its own Fourier transform, and that

$$\text{Fourier transform of } (v \to \varphi(tvg)) = (v \to t^{-2}\det(g)^{-1} \cdot \varphi(t^{-1}v\,{}^\top g^{-1}))$$

where ${}^\top g$ is g-transpose. Then Poisson summation asserts

$$\Theta(tg) = t^{-2}\,\det(g)^{-1} \cdot \Theta(t^{-1}\,{}^\top g^{-1})$$

so then the slight modification for the kernel gives

$$\Theta(tg) - 1 = t^{-2}\,\det(g)^{-1} \cdot [\Theta(t^{-1}\,{}^\top g^{-1}) - 1] + t^{-2}\,\det(g)^{-1} - 1$$

Then we tranform the integral from 0 to 1: at first only for $\mathrm{Re}(s) > 1$ we have

$$\int_0^1 t^{2s}\,(\Theta(tg) - 1)\,\frac{dt}{t} = \int_0^1 t^{2s}\,\Big(t^{-2}\,\det(g)^{-1} \cdot [\Theta(t^{-1}\,{}^\top g^{-1}) - 1] + t^{-2}\,\det(g)^{-1} - 1\Big)\,\frac{dt}{t}$$

Replacing t by $1/t$ turns this into

$$\int_1^\infty t^{-2s}\,\Big(t^2\,\det(g)^{-1} \cdot [\Theta(t\,{}^\top g^{-1}) - 1] + t^2\,\det(g)^{-1} - 1\Big)\,\frac{dt}{t}$$

Explicitly evaluating the last two elementary integrals of powers of t from 1 to ∞, using $\mathrm{Re}(s) > 1$, this is

$$\det(g)^{-1}\int_1^\infty t^{2-2s}\,(\Theta(t\,{}^\top g^{-1}) - 1)\,\frac{dt}{t} + \frac{\det(g)^{-1}}{2s-2} - \frac{1}{2s}$$

Use the fact that g has determinant 1 to simplify this to

$$\int_1^\infty t^{2-2s}\,(\Theta(t\,{}^\top g^{-1}) - 1)\,\frac{dt}{t} + \frac{1}{2s-2} - \frac{1}{2s}$$

Further, for g in $SL(2)$,

$${}^\top g^{-1} = wgw^{-1}$$

where w is the *long Weyl element*

$$w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Since $\mathbb{Z}^2 - (0,0)$ is stable under w, and since the length function $v \to |v|^2$ is invariant under w, we have

$$\Theta(g) = \Theta(wg) = \Theta(gw^{-1})$$

so

$$\Theta({}^\top g^{-1}) = \Theta(g)$$

(This is certainly special to $SL(2,\mathbb{Z})$, as opposed to smaller congruence subgroups.) Thus, the original integral from 0 to 1 becomes

$$\int_1^\infty t^{2-2s}\,(\Theta(tg) - 1)\,\frac{dt}{t} + \frac{1}{2s-2} - \frac{1}{2s}$$

and the *whole* equality, with g of the special form above, is

$$\frac{1}{2}\pi^{-s}\Gamma(s)\,\tilde{E}_s(z) = \int_1^\infty t^{2s}\,(\Theta(tg)-1)\,\frac{dt}{t} + \int_1^\infty t^{2-2s}\,(\Theta(tg)-1)\,\frac{dt}{t} + \frac{1}{2s-2} - \frac{1}{2s}$$

or (multiplying through by 2)

$$\pi^{-s}\Gamma(s)\,\tilde{E}_s(z) = 2\int_1^\infty t^{2s}\,(\Theta(tg)-1)\,\frac{dt}{t} + 2\int_1^\infty t^{2-2s}\,(\Theta(tg)-1)\,\frac{dt}{t} - \frac{1}{1-s} - \frac{1}{s}$$

The integral from 1 to ∞ is nicely convergent for all $s \in \mathbb{C}$, uniformly in g in compacta. And of course the elementary rational expressions of s have meromorphic continuations. Thus, the right-hand side gives a meromorphic continuation of the Eisenstein series. Further, the right-hand side is visibly invariant under $s \to 1-s$.

Finally, it is also visible that the only poles are at $s = 1, 0$, that the residue at $s = 1$ is the constant function 1, and at $s = 0$ the residue is the constant function 0. At $s = 1$ the factor $\pi^{-s}\Gamma(s)$ is holomorphic and has value $1/\pi$, so

$$\mathrm{Res}_{s=1}\,\tilde{E}_s(z) = \pi$$

At $s = 0$ the factor $\pi^{-s}\Gamma(s)$ has a simple pole with residue 1, so $\tilde{E}_s(z)$ itself is holomorphic at $s = 0$, and is the constant function 1.

Now we recover the assertions for $E_s(z)$ itself. The convergence of the infinite product

$$\zeta(2s) = \sum_n \frac{1}{n^{2s}} = \prod_{p\ \mathrm{prime}} \frac{1}{1-p^{-2s}}$$

for $\mathrm{Re}(s) > 1/2$ assures that $\zeta(2s)$ is not zero for $\mathrm{Re}(s) > 1/2$. And $\zeta(2) = \pi^2/6$. These standard facts and the previous discussion give the full result. ///
