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The Constant Term

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• Basic estimates
• The hierarchy of constant terms

Let $G$ be a reductive real Lie group, for example $G = GL(n, \mathbb{R})$. Let $A$ be a maximal $\mathbb{R}$-split torus, in the case of $GL(n, \mathbb{R})$ the diagonal matrices, with connected component of the identity $A^+$. Let $K$ be a maximal compact subgroup of $G$, for $GL(n, \mathbb{R})$ the standard orthogonal group $O(n)$. Let $N$ be the unipotent radical of a minimal parabolic containing $A$, in the case of $GL(n, \mathbb{R})$ upper-triangular unipotent matrices. The Iwasawa decomposition of $G$ is with respect to this data is

$$G = N \cdot A^+ \cdot K$$

Thus, the function $g \mapsto a_g$ defined by expressing $g = na_gk$ with $n \in N$, $a \in A^+$, $k \in K$ is well-defined.

Let $\log : A^+ \to a$ be the inverse of the Lie exponential map from the Lie algebra $a$ of $A^+$ to $A^+$ itself. For $\lambda$ in the complexification $a^* \otimes_{\mathbb{R}} \mathbb{C}$ of the group of characters of $a$, keeping in mind that $a_g \in A^+$, write

$$a^\lambda g = e^{\lambda (\log a_g)}$$

For brevity, we may abbreviate the function $g \mapsto a^\lambda_g$ simply as $a^\lambda$.

[0.1] Lemma: Let $C$ be a compact set in $G$, $x \in G$. Then there is a compact subset $C_A$ of $A^+$ such that $y \in xC$ implies $a_y \in a_x \cdot C_A$.

Proof: Let $G = N \cdot A^+ \cdot K$ be an Iwasawa decomposition as above. Given a compact subset $C$ of $G$, $C \cdot K$ is still compact and contains $C$, and is right $K$-stable. For a right $K$-stable compact subset $C$

$$C \subset (N A^+ \cap C) \cdot K$$

since in Iwasawa coordinates $pk \in C$ with $p \in NA^+$ and $k \in K$ implies by right $K$-stability that $p = (pk) \cdot k^{-1}$ is also in $C$. There are compact subsets $C_N \subset N$, $C_A \subset A^+$ so that

$$K \cdot C \subset C_N \cdot C_A \cdot K$$

Then

$$xC \subset Na_xK \cdot C \subset Na_x \cdot C_N C_A K \subset N \cdot (a_x N a_x^{-1}) \cdot (a_x C_A) \cdot K \subset N \cdot (a_x C_A) \cdot K$$

which shows that for $y \in xC$ the element $a_y$ is in $a_x C_A$. //

A left $N \cap \Gamma$-invariant function $\mathbb{C}$-valued $f$ on $G$ is said to be of moderate growth of exponent $\lambda$ on a fixed Siegel set

$$S_t = \{ x \in G : a^\lambda_x \geq t \text{ for all positive simple roots } \alpha \}$$

if

$$f(g) = O(a^\lambda_g) \quad (\text{for } g \in S_t)$$

[0.2] Corollary: Fix an exponent $\lambda$. For any $\varphi \in C_c^\infty(G)$ there is a constant $c$ and constant $0 < \mu$ so that, for any $f$ of moderate growth of exponent $\lambda$ on a Siegel set $S_t$, $\varphi \cdot f$ is of moderate growth of exponent $\lambda$ on the Siegel set $S_{\mu t}$.
The interchange of differentiation and integration is justified by observing that the integral is compactly supported, continuous, and takes values in a quasi-complete locally convex topological vector space on which differentiation is a continuous linear map.
Remark: For $G = GL(n)$, the standard simple positive roots are
\[
\alpha_i \left( \begin{array}{c} m_1 \\ m_2 \\ \vdots \\ m_n \end{array} \right) = m_i/m_{i+1}
\]
for $1 \leq i \leq n - 1$.

Remark: For non-maximal parabolics there is not the same sort of clear decrease of the exponent of growth. Instead, a somewhat more complicated estimate holds.

Proof: First, we give a proof for $G = GL(2)$. Normalizing the measure of $(\Gamma \cap N) \setminus N$ to be 1,
\[
(f_P - f)(x) = \int_{(\Gamma \cap N) \setminus N} f(nx) - f(x) \, dn = \int_{0 \leq t \leq 1} f(e^{tX} \cdot x) - f(x) \, dt
\]
where $X$ is the element
\[
X = \left( \begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array} \right)
\]
in the Lie algebra of $N$. By the fundamental theorem of calculus
\[
f(e^{tX} \cdot x) - f(x) = \int_0^t \frac{\partial}{\partial r} \big|_{r=0} f(e^{(r+s)X} \cdot x) \, ds = \int_0^t -X^{\text{left}} f(e^{sX} \cdot x) \, ds
\]
where $X^{\text{left}}$ is the natural right-$G$-invariant operator attached to $X$ via the left regular representation. The main mechanism of this proof resides in the conversion of this operator to a left-$G$-invariant operator attached to the right regular representation, as follows.

\[
X^{\text{left}} f(e^{sX} \cdot x) = \left( \frac{\partial}{\partial r} \big|_{r=0} f \right) (e^{sX} \cdot x) = \left. \frac{\partial}{\partial r} \right|_{r=0} f(e^{sX} \cdot x \cdot e^{r \cdot \text{Ad}^{-1}(X)}) = \text{Ad}^{-1}(X) f(e^{sX} \cdot x)
\]
where $\text{Ad}^{-1}(X)$ is the left-$G$-invariant operator attached to $X$ via the right regular representation. Let
\[
x = n_x a_x \theta_x
\]
with $n_x \in N$, $a_x \in M$, $\theta_x \in K$. Then
\[
\text{Ad}^{-1}(X) = \text{Ad} (\theta_x^{-1} a_x^{-1} n_x^{-1})(X) = \text{Ad} (\theta_x^{-1} a_x^{-1})(X)
\]
Further,
\[
\text{Ad} a_x^{-1}(X) = (a_x)^{-2} \cdot X
\]
since $X$ is in the $a_x \rightarrow a_x^2$ rootspace. Then
\[
\text{Ad} (\theta_x^{-1} a_x^{-1})(X) = a_x^{-2} \cdot \text{Ad} \theta_x^{-1}(X) = a_x^{-2} \cdot \sum_i c_i(\theta_x)Y_i
\]
where the $c_i$ are continuous functions (depending upon $X$) on $K$ and $\{Y_i\}$ is a basis for the Lie algebra of $G$. Since the $c_i$ are continuous on the compact set $K$, they have a uniform bound $c$. Altogether,
\[
(f_P - f)(x) = \int_{0 \leq t \leq 1} \int_{0 \leq s \leq t} a_x^{-2} \left( -\sum_i c_i(\theta_x)Y_i \right) f(e^{sX} \cdot x) \, ds \, dt
\]
\[
= a_x^{-2} \cdot \sum_i c_i(\theta_x) \int_{0 \leq t \leq 1} \int_{0 \leq s \leq t} (-X_i f)(e^{sX} \cdot x) \, ds \, dt = a_x^{-2} \cdot \sum_i c_i(\theta_x) \int_{0 \leq t \leq 1} (-Y_i f)(e^{tX} \cdot x) \, dt \\
= a_x^{-2} \cdot \sum_i c_i(\theta_x)(-Y_i f)p(x)
\]

In this case the only root in \( N \) is \( a_x \rightarrow a_x^2 \), so the assertion of the proposition holds in this case (where \( G = GL(2) \)).

Next, we redo the proof to work at least for maximal proper parabolics \( P \) having abelian unipotent radicals \( N \). (The general case is complicated only in aspects somewhat irrelevant to the main point.) Normalizing the measure of \((\Gamma \cap N)\backslash N\) to be 1, we can write

\[
(f - f_P)(x) = \int_{(\Gamma \cap N)\backslash N} f(nx) - f(x) \, dn = \int_{[0,1]^k} f(e^{t_1X_1 + \cdots + t_kX_k} \cdot x) - f(x) \, dt_1 \cdots dt_k
\]

where \( X_1, \ldots, X_k \) is a basis for the Lie algebra of \( N \) so that

\[
\{t_1X_1 + \cdots + t_kX_k : 0 \leq t_i \leq 1, \ 1 \leq i \leq k\}
\]

maps bijectively to \((\Gamma \cap N)\backslash N\), using the abelian-ness to know that this is possible.

By the fundamental theorem of calculus, for \( X \) in the Lie algebra,

\[
f(e^{tX} \cdot x) - f(x) = \int_0^t \frac{\partial}{\partial r} |_{r=0} f(e^{(r+s)X} \cdot x) \, ds = \int_0^t -X_{left} f(e^{sX} \cdot x) \, ds
\]

where \( X_{left} \) is the natural right-\(G\)-invariant operator attached to \( X \). (The main mechanism of this proof resides in the conversion of such operators to left-\(G\)-invariant operators.) Rewrite this integral (by unteleoping) as a sum of \( k \) integrals of the form

\[
\int_{[0,1]^k} f(e^{t_1X_1 + \cdots + t_iX_i} \cdot x) - f(e^{t_1X_1 + \cdots + t_{i-1}X_{i-1}} \cdot x) \, dt_1 \cdots dt_k
\]

Fix the index \( i \), and abbreviate

\[
Y = t_1X_1 + \cdots + t_{i-1}X_{i-1}
\]

and let \( t = t_i \), \( X = X_i \). Then, by the fundamental theorem of calculus, the previous integrand integrated just in \( t = t_i \) is

\[
\int_{0 \leq t \leq 1} f(e^{Y+tX} \cdot x) - f(e^{Y} \cdot x) \, dt = \int_{0 \leq t \leq 1} \int_{0 \leq s \leq t} \frac{\partial}{\partial s} |_{s=0} f(e^{Y+sX+tX} \cdot x) \, ds \, dt
\]

\[
= \int_{0 \leq t \leq 1} \int_{0 \leq s \leq t} (-X_{left} f)(e^{Y+sX} \cdot x) \, ds \, dt
\]

We convert the operator \( X_{left} \) to an operator on the right (and, thus, left \( G \)-invariant), as follows.

\[
(X_{left} f)(e^{Y+sX} \cdot x) = \left(\frac{\partial}{\partial r} |_{r=0} f\right)(e^{Y+rX+sX} \cdot x)
\]

\[
= \frac{\partial}{\partial r} |_{r=0} f\left(e^{Y+sX} \cdot x \cdot e^{r \cdot \text{Ad} x^{-1}(X)}\right) = \text{Ad} x^{-1}(Y) f(e^{Y+sX} \cdot x)
\]

where \( \text{Ad} x^{-1}(X) \) is the left-\(G\)-invariant operator attached to \( Y \) via the right regular representation. Let

\[
x = n_x a_x \theta_x
\]
with $n_x \in N$, $a_x \in M$, $\theta_x \in K$. Then

$$\text{Ad} x^{-1}(X) = \text{Ad} (\theta_x^{-1} a_x^{-1} n_x^{-1})(X) = \text{Ad} (\theta_x^{-1} a_x^{-1})(X)$$

using again the assumed abelian-ness of the Lie algebra of $N$. Now suppose further that $X$ lies in the $\beta$ rootspace in the Lie algebra of $N$. Then

$$\text{Ad} a_x^{-1}(X) = \beta(a_x)^{-1} \cdot X$$

and

$$\text{Ad} (\theta_x^{-1} a_x^{-1})(X) = \beta(a_x)^{-1} \cdot \text{Ad} \theta_x^{-1}(X) = \beta(a_x)^{-1} \cdot \sum_{1 \leq i \leq k} c_i(\theta_x) Y_i$$

where the $c_i$ are continuous functions (depending upon $X$) on $K$ and $\{ Y_i \}$ is a basis for the Lie algebra of $G$. Since the $c_i$ are continuous on the compact set $K$, they have a uniform bound $c$ (depending on $X$). Then altogether

$$\int_{0 \leq t \leq 1} f(e^{Y+tx} \cdot x) - f(e^Y \cdot x) \, dt = \beta(a_x)^{-1} \cdot \sum_{1 \leq i \leq k} c_i(\theta_x) \int_{0 \leq t \leq 1} \int_{0 \leq s \leq t} (-Y_i f)(e^{Y+sx} \cdot x) \, ds \, dt$$

On Siegel sets, for all such $\beta$,

$$\beta(a_x)^{-1} = O(a_x^{-\alpha})$$

Thus, using the exponent $\lambda$ moderate growth of each of the functions $Y_i f$, we have found

$$\int_{0 \leq t \leq 1} f(e^{Y+tx} \cdot x) - f(e^Y \cdot x) \, dt = O(a_x^{\lambda-\alpha})$$

or, in the original notation,

$$\int_{0 \leq t \leq 1} f(e^{t_1 X_1 + \cdots + t_i \cdot x_i} \cdot x) - f(e^{t_1 X_1 + \cdots + t_i x_i} \cdot x) \, dt_i = O(a_x^{\lambda-\alpha})$$

Then, integrating in $dt_1, \ldots, dt_{i-1}$ and in $dt_{i+1}, \ldots, dt_k$ over copies of $[0,1]$ gives the same estimate for the $k$-fold integral:

$$\int_{[0,1]^k} f(e^{t_1 X_1 + \cdots + t_i \cdot x_i} \cdot x) - f(e^{t_1 X_1 + \cdots + t_i x_i} \cdot x) \, dt_1 \ldots dt_k = O(a_x^{\lambda-\alpha})$$

This is the assertion. ///

[0.7] Corollary: Let $P$ be a maximal proper parabolic, with $\alpha$ the unique simple positive root in $N$. For $f$ smooth of moderate growth of exponent $\lambda$ in Siegel sets, and for $\varphi : f = f$ for some $\varphi \in C^\infty_c(G)$, $f - f_P$ is of exponent $\lambda - \ell \alpha$ for all positive integers $\ell$.

Proof: If $\varphi f = f$ then the previous corollary on uniform moderate growth asserts that $Lf$ is of moderate growth exponent $\lambda$ for every $L$ in the universal enveloping algebra. On the other hand, the previous proposition shows that since every $X f$ is of exponent $\lambda$, $f - f_P$ is of exponent $\lambda - \alpha$. But then the uniform moderate growth assures that every $X(f - f_P)$ is of exponent $\lambda - \alpha$, as well. Applying the last proposition again, we find that

$$(X f - X f_P) - (X f - X f_P)_P = X f - X f_P = X(f - f_P)$$

is of exponent $\lambda - 2 \cdot \alpha$. This begins an induction which proves the corollary. ///
1. The hierarchy of constant terms

Let $\Delta$ denote the collection of simple (positive) roots. For each $\alpha \in \Delta$, there is a maximal proper parabolic $P_\alpha$ whose unipotent radical $N_\alpha$ has Lie algebra $n$ containing the $\alpha^{th}$ root space $g_\alpha$ in the Lie algebra $g$ of $G$. In particular, the Lie algebra $n$ is exactly the sum of all the rootspaces $g_\beta$ with $\beta \geq \alpha$.

Let $c_\alpha$ be the mapping which computes the $P_\alpha$ constant term

$$c_\alpha f(g) = \int_{\Gamma N_\alpha \backslash N_\alpha} f(ng) \, dn$$

for locally integrable $f$ left-invariant under a co-compact subgroup $\Gamma_{N_\alpha}$ of $N_\alpha$. The group $N_\alpha$ has Haar measure normalized so that $\text{meas}(\Gamma_{N_\alpha} \backslash N_\alpha) = 1$.

In particular, for simplicity we assume a consistency relation among these co-compact subgroups $\Gamma_{N_\alpha}$ by letting $\Gamma_{N_{\min}}$ be a cocompact subgroup of the unipotent radical of a minimal parabolic

$$P_{\min} = \cap_{\alpha \in \Delta} P_\alpha$$

and take

$$\Gamma_{N_\alpha} = N_\alpha \cap \Gamma_{N_{\min}}$$

A simple example is to take $G = GL(n, \mathbb{R})$ and

$$\Gamma_{N_{\min}} = \text{upper-triangular unipotent matrices with integer entries}$$

[1.1] **Lemma:** For simple roots $\alpha, \beta$,

$$c_\alpha \circ c_\beta = c_\beta \circ c_\alpha$$

**Proof:** A direct computation, changing variables in the integrals definitions of these operators, using the unimodularity of the groups, etc. ///

[1.2] **Proposition:** Let $P_S$ be the parabolic whose unipotent radical contains exactly the simple roots $S$. Let $c_P$ be the constant term operator for $P$. Then

$$1 - c_P = \prod_{\alpha \in S} (1 - c_\alpha)$$

[1.3] **Corollary:** Let $f$ be left $\Gamma_{N_{\min}}$-invariant and $Z$-finite and $K$-finite. Then

$$\left( \prod_{\alpha \in \Delta} (1 - c_\alpha) \right) f$$

is of rapid decay in any Siegel set aligned with the implied family of parabolic subgroups.