Distribution $|\det x|^s$ on $p$-adic matrices

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Let $F$ be a $p$-adic field with integers $\mathfrak{o}$, local parameter $\varpi$, and residue field cardinality $q$. Let $A = M_n(F)$ be the $F$-vectorspace of $n$-by-$n$ matrices over $k$, and $G = G\text{L}_n(F)$. Let $u_s$ be the tempered distribution

$$u_s(f) = \int_G |\det x|^s f(x) \, d^x x$$

(Schwartz function $f$ on $A$, for $\text{Re}(s) \gg 1$)

where $d^x x$ denotes a Haar measure on $G$. Up to constants, for (additive) Haar measure $d^x x$ on $A$, $d^x x = d^+ x / |\det x|^n$. For brevity, write $|x|$ for $|\det x|$ when possible.

[0.1] Convergence  The integral defining $u_s$ converges absolutely in $\text{Re}(s) > n - 1$:

Recall the Iwasawa decomposition $G = P \cdot K$ with $P$ the parabolic subgroup of upper-triangular matrices. Since $K$ is open in $G$, Haar measure on $G$ restricted to $K$ is Haar measure on $K$. Recall the integral formula

$$\int_G f(g) \, dg = \int_K \int_P f(pk) \, dp \, dk \quad \text{(up to normalization, with left Haar measure on $P$)}$$

In Levi-Malcev coordinates $NM = P$, with $N$ the unipotent radical and $M$ diagonal matrices, up to normalization, left Haar measure on $P$ is

$$d\left(\begin{array}{cccc}\frac{1}{2} & x_{12} & \cdots & x_{1n} \\ & 1 & \cdots & \\ & & \ddots & \cdots \\ & & & 1 \end{array}\right) \left(\begin{array}{c} y_1 \\ & y_2 \\ & & \ddots \\ & & & y_n \end{array}\right)$$

$$= dx_{12} dx_{13} \ldots dx_{n-1,n} \prod_{j} dy_j / |y_1|^{1+(n-1)} |y_2|^{1+(n-3)} |y_3|^{1+(n-5)} \ldots |y_n|^{1-(n-1)}$$

with additive Haar measures in the coordinates. For $\chi$ the characteristic function of $M_n(\mathfrak{o})$, and $\text{Re}(s) \gg 1$, up to normalization constants,

$$u_s(\chi) = \int_G \chi(g) |\det x|^s \, dg = \int_P \chi(p) |\det p|^s \, dp = \prod_j \int_{\mathfrak{o}\setminus k} \prod_{i < j} \left( \int_{y_{ij}^{-1}} \, 1 dx_{ij} \right) |y_j|^{-s} \frac{dy_j}{|y_j|^{1+(n-2j-1)}}$$

$$= \prod_j \int_{\mathfrak{o}\setminus k} |y_j|^{-s} \frac{dy_j}{|y_j|^{1+(n-2j-1)}} = \prod_j \int_{\mathfrak{o}\setminus k} 1 dx_{ij} = \prod_j 1 - q^{-(s-n)}$$

Thus, $u_s(\chi)$ converges absolutely in $\text{Re}(s) > n - 1$, admits a meromorphic continuation, and definitely blows up as $s \to (n-1)^+$. Using the homogeneity of $u_s$, the integral expression for $u_s(f)$ for any Schwartz function $f$ is dominated by the integral for $u_s(\chi)$, so $u_s$ gives a tempered distribution in $\text{Re}(s) > n - 1$.

[0.2] Meromorphic continuation and residues of $u_s(\chi)$  The outcome of the computation of $u_s(\chi)$ to understand convergence also gives a meromorphic continuation in $s$, with simple poles (and non-zero residues) at $s = 1, 0$ (and at points differing by integer multiples of $2\pi i / \log q$ from these).

[0.3] Meromorphic continuation of $u_s(f)$  For an arbitrary Schwartz function $f$ the value $u_s(f)$ can be meromorphically continued similarly, as follows. First, since $u_s$ is right $K$-invariant, first average $f$ on the right over $K$, and then use a Levi-Malcev decomposition:

$$\int_G |\det x|^s f(x) \, d^x x = \int_P |\det p|^s \left( \int_K f(xk) \, dk \right) \, dp = \int |m_1|^{s-(n-1)} \ldots |m_n|^{s-(n-1)} f^K(nm) \, dn \, dm$$
where $f^K$ is the averaged $f$. Since $f^K$ is itself a Schwartz function, it is a finite linear combination of monomials

$$\varphi(x) = \prod_{ij} \varphi_{ij}(x_{ij})$$

of Schwartz functions $\varphi_{ij}$ in the coordinates $x_{ij}$. Of course, the support of $\varphi_{ij}$ must include 0 for $i < j$, or else $\varphi(p) = 0$. For $i < j$, the relevant integral is

$$\int_F \varphi_{ij}(x_{ij} m_{ij}) \, dx_{ij} = |m_{ij}|^{-1} \int_F \varphi_{ij}(x_{ij}) \, dx_{ij}$$

The whole is

$$\prod_{i>j} \varphi_{ij}(0) \times \prod_{1<i<j} \int_F \varphi_{ij}(x_{ij}) \, dx_{ij} \times \prod_{i} \int_{F^\times} \varphi_{ii}(m_{ii}) \frac{|m_{ii}|^{s-(n-2)i}}{|t|^s} \, d^\times m_{ii}$$

$$= \prod_{i>j} \varphi_{ij}(0) \times \prod_{1<i<j} \int_F \varphi_{ij}(x_{ij}) \, dx_{ij} \times \prod_{i} \int_{F^\times} \varphi_{ii}(t) \frac{|t|^{s-n+i}}{t} \, d^\times t$$

The first two products are constants. Each integral in the last product is an Iwasawa-Tate local zeta integral: when the support of $\varphi_{ii}$ does not include 0, it is a polynomial in $q^{-s}$, and when the support of $\varphi_{ii}$ includes 0, the zeta integral is a sum of a polynomial in $q^{-s}$ and a constant multiple of $\frac{1}{1-q^{s-n+i}}$.

Finite sums of such expressions admit meromorphic continuations with poles at most at $s = n - 1, n - 2, \ldots, 2, 1, 0$ (and points differing from these by integer multiples of $2\pi i/\log q$). Poles, if any, are simple.

Thus, $v_s(f) = (1 - q^{-s}) \ldots (1 - q^{-s}) \cdot u_s(f)$ has a meromorphic for every Schwartz function $f$. That is, $v_s$ is weakly holomorphic. Weak holomorphy implies (strong) holomorphy for vector-valued functions with values in a quasi-complete locally convex topological vector space. Tempered distributions are such. Thus, $v_s$ is a holomorphic tempered-distribution-valued function of $s \in \mathbb{C}$. In particular, the residues of $u_s$ at poles are tempered distributions.

**0.4 Support of residues** For $f$ a Schwartz function with support inside $G$, the meromorphic scalar-valued function $u_s(f)$ is **entire**. Thus, the residues of $u_s$ at $s = n - 1, n - 2, \ldots, 0$ are tempered distributions supported on the set $A^{<n}$ of matrices of less-than-full rank.

**0.5 Uniqueness and existence of equivariant distributions** The standard argument shows that, given $s \in \mathbb{C}$, there is a unique tempered distribution on $G$ such that $u(AB) = |\det A \cdot \det B|^s \cdot u(x)$, since $G$ acts transitively on itself. The tempered distribution is given by the integral for $u_s$ in the range of convergence, and by meromorphic continuation otherwise.

Let

$$G^1 = \{g \in G : |\det g| = 1\}$$

The product $G^1 \times G^1 K$ acts transitively on the set $A_r$ of matrices of a given rank $r < n$ by $(g \times h)(x) = g^{-1} x h$. The isotropy group of

$$E_r = \begin{pmatrix} 1_r & 0 \\ 0 & 0_{n-r} \end{pmatrix}$$

is

$$H_r = \{ \begin{pmatrix} A & * \\ 0 & D \end{pmatrix} \times \begin{pmatrix} a & 0 \\ c & D \end{pmatrix} : D \in GL_{n-r}(\mathbb{O}), \ a, A \in GL_r(F), \ |\det a| = |\det A| = |\det D|^{-1} \} \subset G^1 \times G^1$$

Both $H_r$ and $G^1 \times G^1$ are **unimodular**, so there is a unique $G^1 \times G^1$-invariant measure on $A_r \approx G^1 \times G^1 / H_r$, and integration against this measure gives the unique $G^1 \times G^1$-invariant distribution on Schwartz functions supported on the set $A^{2r}$ of matrices of rank $\geq r$. 

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At the same time, $G^1 \times K$ is already transitive on $A_r$, so up to scalars there is unique $G^1 \times K$-invariant measure and corresponding distribution. We can easily write a formula for it in terms of Euclidean coordinates, namely

$$u^{(r)}(f) = \int_K \int_{F \times G} \int_{F \times (n-r) \times G} f\left( \begin{pmatrix} x_{11} & 0 \\ x_{21} & 0_{n-r} \end{pmatrix} \right) \, dx_{21} \, dx_{11} \, dk$$

By the uniqueness of $G^1 \times K$-invariant functional, this integral formula must also be a $G^1 \times G^1$-invariant functional. Similarly, the $K \times G^1$-invariant form of that integral must give the same functional:

$$u^{(r)}(f) = \int_K \int_{F \times G} \int_{F \times (n-r) \times G} f(k \cdot \begin{pmatrix} x_{11} & x_{12} \\ 0 & 0_{n-r} \end{pmatrix} \cdot k) \, dx_{12} \, dx_{11} \, dk$$

Equality up to constants follows from uniqueness, and the constant is 1 because the two integrals agree on the characteristic function of $\Lambda$.

These integrals converge absolutely, so extend to tempered distributions on the whole Schwartz space. Changing variables in the first integral expression, the equivariance under the full group $G = GL_n$ is

$$u^{(r)}\left( x \to f(Ax) \right) = |\det A|^{k-r} \cdot u^{(r)}(f) \quad (\text{for } A \in GL_n(F))$$

Changing variables in the second integral expression,

$$u^{(r)}\left( x \to f(xB) \right) = |\det B|^{k-r} \cdot u^{(r)}(f) \quad (\text{for } B \in GL_n(F))$$

The residue of $u_s$ at $s = r < n$ is supported on the set $A^{\leq r}$ of matrices of rank $\leq r$, and has the same equivariance under $G \times G$ as does $u^{(r)}$. Suggesting that, up to a constant, the distribution $u^{(r)}$ is the residue of $u_s$ at $s = r < n$.

Indeed, on Schwartz functions supported on $A^{\leq r}$, the uniqueness result just above does show that the residue of $u_s$ at $r$ is a constant multiple of $u^{(r)}$.

The integral expression for $u^{(r)}$ specifies it on the whole Schwartz space. The appearance of the residue as a residue specifies it on the whole Schwartz space. The difference $v$ of suitable multiples vanishes on Schwartz functions supported on $A^{2r}$. This difference restricted to Schwartz functions supported on $A^{2r-1}$ is $G^1 \times G^1$-invariant, so must be a multiple of $u^{(r-1)}$. However, the $G \times G$-equivariance does not match that of $u^{(r-1)}$, so this restriction to $A^{2r-1}$ is 0. Similarly, the restriction of $v$ to Schwartz functions supported on $A^{2r-2}$ must be a multiple of $u^{(r)}$, and the equivariance forces it to be 0. Continuing, we find that the residue of $u_s$ at $s = r < n$ is a multiple of $u^{(r)}$.

**[0.6] Non-extendability of $|\det x|^1$ for $GL_2$** In the small example $G = GL_2(F)$, a relatively elementary argument shows that $u_1(f) = \int_G |x|^1 f(x) \, dx$ has no extension from Schwartz functions supported on $G$ to the whole space of Schwartz functions on $A$. Specifically, we claim that any tempered distribution $u$ with the homogeneity property

$$u(R_g f) = |\det g|^{-1} \cdot u(f) \quad (\text{with } (R_g f)(x) = f(xg))$$

is supported on $A^{\leq 1}$. This will follow from the Hecke operator identity (proven below)

$$\text{ch}_K = \text{ch}_\Lambda + q \cdot \text{ch}_{\varpi \Lambda} - T_p(\text{ch}_\Lambda)$$

where $\text{ch}_X$ is the characteristic function of a set, $\Lambda = M_2(\mathfrak{o})$, and $T_p$ is the Hecke operator incarnated as

$$(T_p f)(x) = \int_G \text{ch}_{D_1}(g^{-1}) \cdot f(xg) \, dg \quad (\text{with } D_n = \{ g \in M_2(\mathfrak{o}), |\det g| = q^{-n} \})$$
Indeed, a tempered distribution \( u \) with the indicated homogeneity property restricted to Schwartz functions on \( G \) is \( c \cdot u_1 \) for some constant \( c \), by uniqueness. We show that \( c = 0 \). The interaction of Hecke operator and \( u \) is easily determined:

\[
u(T_p f) = u \left( \int_G \text{ch}_{D_1}(g^{-1}) \cdot R_g f \, dg \right) = \int_G \text{ch}_{D_1}(g^{-1}) \cdot u(R_g f) \, dg = \int_G \text{ch}_{D_1}(g) \cdot u(f) \cdot |\det g|^{-1} \, dg
\]

\[
= u(f) \cdot q^{-1} \cdot \int_G \text{ch}_{D_1}(g) \, dg = u(f) \cdot q^{-1}(q + 1)
\]

Applying \( u \) to the identity of characteristic functions gives

\[
u(\text{ch}_K) = \nu(\text{ch}_A) + q \cdot \nu(\text{ch}_\varpi \Lambda) - u(T_p(\text{ch}_\Lambda))
\]

and then

\[
c \cdot u_1(\text{ch}_K) = \nu(\text{ch}_A) + q \cdot q^{-2} \cdot \nu(\text{ch}_\Lambda) - q^{-1}(q + 1) \nu(\text{ch}_\Lambda)
\]

\[
= \nu(\text{ch}_\Lambda) \cdot \left( 1 + q \cdot q^{-2} - q^{-1}(q + 1) \right) = \nu(\text{ch}_\Lambda) \cdot 0
\]

yielding \( c = 0 \). To prove the identity

\[
\nu(\text{ch}_K) = \nu(\text{ch}_A) + q \cdot \text{ch}_\varpi \Lambda - T_p(\text{ch}_\Lambda)
\]

we explicate the action of \( T_p \) on the functions \( \text{ch}_{D_n} \), since

\[
\text{ch}_{A} = \sum_{n \geq 0} \text{ch}_{D_n} + \text{ch}_{A^{\leq 1}}
\]

As in the classical context,

\[
(T_p \text{ch}_{D_n})(x) = \int_G \text{ch}_{D_1}(g^{-1}) \cdot \text{ch}_{D_n}(xg) \, dg
\]

is certainly 0 unless \( x \in D_{n+1} \). Left modulo \( K \), \( D_{n+1} \) has representatives

\[
\begin{pmatrix}
\varpi^i & b \\
0 & \varpi^{n+1-i}
\end{pmatrix}
\]

(with \( b \) mod \( \varpi^{n+1-i} \))

Similarly for \( g \) right modulo \( K \), equivalently, for \( g^{-1} \in D_1 \) left modulo \( K \), take representatives

\[
\begin{pmatrix}
\varpi^{-1} & 0 \\
0 & 1
\end{pmatrix}, \begin{pmatrix} 1 & b' \\ 0 & \varpi \end{pmatrix}
\]

(with \( b' \) mod \( \varpi \))

and then

\[
\begin{pmatrix}
\varpi^{-1} & 0 \\
0 & 1
\end{pmatrix}, \begin{pmatrix} 1 & -b'/\varpi \\ 0 & 1/\varpi \end{pmatrix}
\]

(with \( b' \) mod \( \varpi \))

and

\[
xg = \begin{pmatrix}
\varpi^{i-1} & b \\
0 & \varpi^{n+1-i}
\end{pmatrix}, \begin{pmatrix} \varpi^i & b\varpi^{-1} - b'\varpi^{i-1} \\ 0 & \varpi^{n-i} \end{pmatrix}
\]

(with \( 0 \leq i \leq n + 1 \))

Given \( x \), the integral produces the value 1 for \( g \) such that \( xg \in M_2(\alpha) \). The first case gives a value 1 exactly for \( i \geq 1 \). The second family gives 1 for \( 1 \leq i \leq n \) and \( b \in \varpi \vartheta \), or for \( i = 0 \) and \( b = b' \) mod \( \varpi \). In summary,

\[
T_p \text{ch}_{D_n} \begin{pmatrix}
\varpi^i & b \\
0 & \varpi^{n+1-i}
\end{pmatrix} = \begin{cases}
1 + q & \text{for } 1 \leq i \leq n \text{ and } b \in \varpi \vartheta \\
1 & \text{for } 1 \leq i \leq n \text{ and } b \notin \varpi \vartheta \\
1 & \text{for } i = 0 \\
1 & \text{for } i = n + 1
\end{cases}
\]

\[(0 \leq i \leq n + 1 \text{ and } b \mod \varpi^{n+1-i})\]
That is,

\[ T_p \text{ch}_{D_n} = \begin{cases} 
q \cdot \text{ch}_{\varpi D_{n-1}} + \text{ch}_{D_{n+1}} & \text{for } n \geq 1 \\
\text{ch}_{D_1} & \text{for } n = 0
\end{cases} \]

Ignoring singular matrices since the outcome \( T_p \text{ch}_\Lambda \) is guaranteed to be a Schwartz function,

\[ T_p \text{ch}_\Lambda = T_p \left( \text{ch}_{D_o} + \sum_{n \geq 1} \text{ch}_{D_n} \right) = \text{ch}_{D_1} + \sum_{n \geq 1} \left( q \cdot \text{ch}_{\varpi D_{n-1}} + \text{ch}_{D_{n+1}} \right) = q \text{ch}_\varpi + \text{ch}_\Lambda - \text{ch}_K \]

since \( D_o = K \). This rearranges to the asserted identity, from which the non-extendability follows.