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Unbounded operators, Friedrichs' extension theorem

Paul Garrett garrett@math.umn.edu http://www.math.umn.edu/˜garrett/

It is amazing that *resolvents* $R_\lambda = (T - \lambda)^{-1}$ exist, as *everywhere-defined*, continuous linear maps on a Hilbert space, even for T *unbounded*, and only densely-defined. Of course, some further hypotheses on T are needed, but these hypotheses are met in useful situations occurring in practice.

In particular, we will need that T is *symmetric*, in the sense that $\langle Tv, w\rangle = \langle v, Tw\rangle$ for v, w in the domain of T. And we will need to replace T by its *Friedrichs extension*, described explicitly below. For example, the Friedrichs extension replaces genuine differentiation by L^2-differentiation. [1]

So-called **unbounded operators** on a Hilbert space V are not literally operators on V, being defined on *proper subspaces* of V. For unbounded operators on V, the actual *domain* is an essential part of a description: an unbounded operator T on V is a subspace D of V and a linear map $T : D \longrightarrow V$. The interesting case is that the domain D is *dense* in V.

The linear map T is most likely *not* continuous when D is given the subspace topology from V, or it would extend by continuity to the closure of D, presumably V.

Explicit naming of the domain of an unbounded operator is often suppressed, instead writing $T_1 \subset T_2$ when T_2 is an **extension** of T_1, in the sense that the domain of T_2 contains that of T_1, and the restriction of T_2 to the domain of T_1 agrees with T_1.

An operator T', D' is a **sub-adjoint** to an operator T, D when

$$\langle Tv, w\rangle \;=\; \langle v, T'w\rangle \qquad\qquad (\text{for } v \in D,\, w \in D')$$

For D *dense*, for given D' there is *at most* one T' meeting the adjointness condition.

The **adjoint** T^* is the *unique maximal* element, in terms of domain, among all sub-adjoints to T. That there is a unique maximal sub-adjoint requires proof, given below.

An operator T is *symmetric* when $T \subset T^*$, and *self-adjoint* when $T = T^*$. These comparisons refer to the *domains* of these not-everywhere-defined operators. In the following claim and its proof, the domain of a map S on V is incorporated in a reference to its *graph*

$$\text{graph } S \;=\; \{v \oplus Sv \,:\, v \in \text{domain } S\} \;\subset\; V \oplus V$$

[0.0.1] Remark: In practice, anticipating that a given unbounded operator is self-adjoint *when extended suitably*, a simple version of the operator is defined on an easily described, small, dense domain, specifying a *symmetric* operator. Then a self-adjoint *extension* is shown to exist, as in Friedrichs' theorem below.

[0.0.2] Remark: A symmetric operator that *fails* to be self-adjoint is necessarily *unbounded*, since bounded symmetric operators are self-adjoint, because of the existence of orthogonal complements in Hilbert spaces. The latter idea is applied to not-necessarily-bounded operators in the following.

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[1] [Friedrichs 1934] construction of suitable extensions predates [Sobolev 1937,1938], though the extensions use an abstracted version of what nowadays are usually called Sobolev spaces. The physical motivation for the construction is *energy estimates*. *Existence* results for self-adjoint extensions had been discussed in [Neumann 1929], [Stone 1929,30,34], but a useful description of a *natural* extension first occurred in [Friedrichs 1934]. Further, a Hilbert-space precursor of the Lax-Milgram theorem of [Lax-Milgram 1954] also appears in [Friedrichs 1934], following by the argument Friedrichs uses to prove that his construction gives an extension.

The direct sum $V \oplus V$ is a Hilbert space, with natural inner product

$$\langle v \oplus w, v' \oplus w' \rangle \;=\; \langle v, v' \rangle + \langle w, w' \rangle$$

Define an isometry U of $V \oplus V$ by

$$U \;:\; V \oplus V \longrightarrow V \oplus V \qquad \text{by} \qquad v \oplus w \longrightarrow -w \oplus v$$

[0.0.3] Claim: Given T with *dense* domain D, there is a unique *maximal* T^*, D^* among all sub-adjoints to T, D. Further, the adjoint T^* is *closed*, in the sense that its *graph* is closed in $V \oplus V$. In fact, the adjoint is *characterized* by its graph, which is the orthogonal complement in $V \oplus V$ to an image of the graph of T, namely,

$$\text{graph } T^* \;=\; \text{orthogonal complement of } U(\text{graph } T)$$

Proof: The adjointness condition $\langle Tv, w \rangle = \langle v, T^*w \rangle$ for given $w \in V$ is an orthogonality condition

$$\langle w \oplus T^*w,\; U(v \oplus Tv) \rangle \;=\; 0 \qquad \text{(for all } v \text{ in the domain of } T)$$

Thus, the graph of *any* sub-adjoint is a subset of

$$X \;=\; U(\text{graph } T)^\perp$$

Since T is densely-defined, for given $w \in V$ there is *at most* one possible value w' such that $w \oplus w' \in X$, so this orthogonality condition determines a well-defined function T^* on a subset of V, by

$$T^*w \;=\; w' \qquad \text{(if there exists } w' \in V \text{ such that } w \oplus w' \in X)$$

The linearity of T^* is immediate. It is maximal among sub-adjoints to T because the graph of any sub-adjoint is a subset of the graph of G^*. Orthogonal complements are closed, so T^* has a closed graph. ///

[0.0.4] Corollary: For $T_1 \subset T_2$ with dense domains, $T_2^* \subset T_1^*$, and $T_1 \subset T_1^{**}$. ///

[0.0.5] Corollary: A self-adjoint operator has a closed graph. ///

[0.0.6] Remark: The closed-ness of the graph of a self-adjoint operator is essential in proving existence of *resolvents*, below.

[0.0.7] Remark: The use of the term *symmetric* in this context is potentially misleading, but standard. The notation $T = T^*$ allows an inattentive reader to forget non-trivial assumptions on the *domains* of the operators. The equality of domains of T and T^* is understandably essential for legitimate computations.

[0.0.8] Proposition: Eigenvalues for symmetric operators T, D are *real*.

Proof: Suppose $0 \neq v \in D$ and $Tv = \lambda v$. Then

$$\lambda \langle v, v \rangle \;=\; \langle \lambda v, v \rangle \;=\; \langle Tv, v \rangle \;=\; \langle v, T^*v \rangle \qquad \text{(because } v \in D \subset D^*)$$

Further, because T^* agrees with T on D,

$$\langle v, T^*v \rangle \;=\; \langle v, \lambda v \rangle \;=\; \overline{\lambda} \bar{v}, v \rangle$$

Thus, λ is real. ///

[0.0.9] Definition: A densely-defined symmetric operator T, D is *positive* (or *non-negative*) when

$$\langle Tv, v\rangle \ \geq \ 0 \qquad\qquad (\text{for all } v \in D)$$

Certainly all the eigenvalues of a positive operator are non-negative real.

[0.0.10] Theorem: (Friedrichs) A *positive*, densely-defined, symmetric operator T, D has a positive *self-adjoint* extension.

Proof: [2] Define a new hermitian form $\langle , \rangle_1$ and corresponding norm $\| \cdot \|_1$ by

$$\langle v, w\rangle_1 \ = \ \langle v, w\rangle + \langle Tv, w\rangle \qquad\qquad (\text{for } v, w \in D)$$

The symmetry and non-negativity of T make this positive-definite hermitian on D. Note that $\langle v, w\rangle_1$ makes sense whenever at least one of v, w is in D.

Let D_1 be the closure in V of D with respect to the metric d_1 induced by $\| \cdot \|_1$. We claim that D_1 is also the d_1-completion of D. Indeed, for v_i a d-Cauchy sequence in D, v_i is Cauchy in V in the original topology, since

$$|v_i - v_j| \ \leq \ |v_i - v_j|_1$$

For two sequences v_i, w_j with the same d-limit v, the d-limit of $v_i - w_i$ is 0. Thus,

$$|v_i - w_i| \ \leq \ |v_i - w_i|_1 \ \longrightarrow \ 0$$

For $h \in V$ and $v \in D_1$, the functional $\lambda_h : v \to \langle v, h\rangle$ has a bound

$$|\lambda_h v| \ \leq \ |v| \cdot |h| \ \leq \ |v|_1 \cdot |h|$$

Thus, the norm of the functional λ_h on D_1 is at most $|h|$. By Riesz-Fischer, there is unique Bh in the Hilbert space D_1 with $|Bh|_1 \leq |h|$, such that

$$\lambda_h v \ = \ \langle Bh, v\rangle_1 \qquad\qquad (\text{for } v \in D_1)$$

Thus,

$$|Bh| \ \leq \ |Bh|_1 \ \leq \ |h|$$

The map $B : V \to D_1$ is verifiably linear. There is an obvious *symmetry* of B:

$$\langle Bv, w\rangle \ = \ \lambda_w Bv \ = \ \langle Bv, Bw\rangle_1 \ = \ \overline{\langle Bw, Bv\rangle_1} \ = \ \overline{\lambda_v Bw} \ = \ \overline{\langle Bw, v\rangle} \ = \ \langle v, Bw\rangle \qquad\qquad (\text{for } v, w \in V)$$

Positivity of B is similar:

$$\langle Bv, v\rangle \ = \ \lambda_v Bv \ = \ \langle Bv, Bv\rangle_1 \ \geq \ \langle Bv, Bv\rangle \geq 0$$

Next, B has dense image in D_1: for $w \in D_1$ such that $\langle Bh, w\rangle_1 = 0$ for all $h \in V$,

$$0 \ = \ \langle w, Bh\rangle \ = \ \lambda_h w \ = \ \langle h, w\rangle \qquad\qquad (\text{for all } h \in V)$$

Thus, $w = 0$, proving density of the image of B in D_1. Finally B is *injective*: if $Bw = 0$, then for all $v \in D_1$

$$0 \ = \ \langle v, 0\rangle_1 \ = \ \langle v, Bw\rangle_1 \ = \ \lambda_w v \ = \ \langle v, w\rangle$$

[2] We essentially follow [Riesz-Nagy 1955], pages 329-334.

Since D_1 is dense in V, $w = 0$. Similarly, if $w \in D_1$ is such that $\lambda_v w = 0$ for all $v \in V$, then $0 = \lambda_w w = \langle w, w \rangle$ gives $w = 0$. Thus, $B : V \to D_1$ is bounded, symmetric, positive, injective, with dense image. In particular, B is self-adjoint.

Thus, B has a possibly *unbounded* positive, symmetric inverse A. Since B injects V to a dense subset D_1, necessarily A *surjects* from its domain (inside D_1) to V. We claim that A is *self-adjoint*. Let $S : V \oplus V \to V \oplus V$ by $S(v \oplus w) = w \oplus v$. Then

$$\text{graph } A \;=\; S(\text{graph } B)$$

Also, in computing orthogonal complements $X^\perp$, clearly

$$(S\,X)^\perp \;=\; S\big(X^\perp\big)$$

From the obvious $U \circ S = -S \circ U$, compute

$$\text{graph } A^* \;=\; (U \text{ graph } A)^\perp \;=\; (U \circ S \text{ graph } B)^\perp \;=\; (-S \circ U \text{ graph } B)^\perp$$

$$=\; -S\big((U \text{ graph } B)^\perp\big) \;=\; -\,\text{graph} A \;=\; \text{graph } A$$

since the domain of B^* is the domain of B. Thus, A is self-adjoint.

We claim that for v in the domain of A, $\langle Av, v \rangle \geq \langle v, v \rangle$. Indeed, letting $v = Bw$,

$$\langle v, Av \rangle \;=\; \langle Bw, w \rangle \;=\; \lambda_w Bw \;=\; \langle Bw, Bw \rangle_1 \;\geq\; \langle Bw, Bw \rangle \;=\; \langle v, v \rangle$$

Similarly, with $v' = Bw'$, and $v \in D_1$,

$$\langle v, Av' \rangle \;=\; \langle v, w' \rangle \;=\; \lambda_{w'} v \;=\; \langle v, Bw' \rangle_1 \;=\; \langle v, v' \rangle_1 \qquad (v \in D_1,\ v' \text{ in the domain of } A)$$

Since B maps V to D_1, the domain of A is contained in D_1. We claim that the domain of A is dense in D_1 in the d-topology, not merely in the subspace topology from V. Indeed, for $v \in D_1$ $\langle , \rangle_1$-orthogonal to the domain of A, for v' in the domain of A, using the previous identity,

$$0 \;=\; \langle v, v' \rangle_1 \;=\; \langle v, Av' \rangle$$

Since B injects V to D_1, A surjects from its domain to V. Thus, $v = 0$.

Last, prove that A is an extension of $S = 1_V + T$. On one hand, as above,

$$\langle v, Sw \rangle \;=\; \lambda_{Sw} v \;=\; \langle v, BSw \rangle_1 \qquad (\text{for } v, w \in D)$$

On the other hand, by definition of $\langle , \rangle_1$,

$$\langle v, Sw \rangle \;=\; \langle v, w \rangle_1 \qquad (\text{for } v, w \in D)$$

Thus,

$$\langle v, w - BSw \rangle_1 \;=\; 0 \qquad (\text{for all } v, w \in D)$$

Since D is d-dense in D_1, $BSw = w$ for $w \in D$. Thus, $w \in D$ is in the range of B, so is in the domain of A, and

$$Aw \;=\; A(BSw) \;=\; Sw$$

Thus, the domain of A contains that of S and extends S. ///

Let $R_\lambda = (T - \lambda)^{-1}$ for $\lambda \in \mathbb{C}$ when this inverse exists as a linear operator defined at least on a dense subset of V.

[0.0.11] Theorem: Let T be self-adjoint and densely defined. For $\lambda \in \mathbb{C}$, $\lambda \notin \mathbb{R}$, the operator R_λ is everywhere defined on V, and the operator norm is estimated by

$$\|R_\lambda\| \;\le\; \frac{1}{|\mathrm{Im}\,\lambda|}$$

For T *positive*, for $\lambda \notin [0, +\infty)$, R_λ is everywhere defined on V, and the operator norm is estimated by

$$\|R_\lambda\| \;\le\; \begin{cases} \dfrac{1}{|\mathrm{Im}\,\lambda|} & (\text{for } \mathrm{Re}\,(\lambda) \le 0) \\ \dfrac{1}{|\lambda|} & (\text{for } \mathrm{Re}\,(\lambda) \ge 0) \end{cases}$$

Proof: For $\lambda = x + iy$ off the real line and v in the domain of T,

$$|(T-\lambda)v|^2 \;=\; |(T+x)v|^2 + \langle (T-x)v, iyv\rangle + \langle iyv, (T-x)v\rangle + y^2|v|^2$$

$$=\; |(T+x)v|^2 - iy\langle (T-x)v, v\rangle + iy\langle v, (T-x)v\rangle + y^2|v|^2$$

The symmetry of T, and the fact that the domain of T^* contains that of T, implies that

$$\langle v, Tv\rangle \;=\; \langle T^*v, v\rangle \;=\; \langle Tv, v\rangle$$

Thus,

$$|(T-\lambda)v|^2 \;=\; |(T-x)v|^2 + y^2|v|^2 \;\ge\; y^2|v|^2$$

Thus, for $y \neq 0$, $(T-\lambda)v \neq 0$. Let D be the domain of T. On $(T-\lambda)D$ there is an inverse R_λ of $T-\lambda$, and for $w = (T-\lambda)v$ with $v \in D$,

$$|w| \;=\; |(T-\lambda)v| \;\ge\; |y|\cdot|v| \;=\; |y|\cdot|R_\lambda(T-\lambda)v| \;=\; |y|\cdot|R_\lambda w|$$

which gives

$$|R_\lambda w| \;\le\; \frac{1}{|\mathrm{Im}\,\lambda|}\cdot|w| \qquad (\text{for } w = (T-\lambda)v,\ v \in D)$$

Thus, the operator norm on $(T-\lambda)D$ satisfies $\|R_\lambda\| \le 1/|\mathrm{Im}\,\lambda|$ as claimed.

We must show that $(T-\lambda)D$ is the whole Hilbert space V. If

$$0 \;=\; \langle (T-\lambda)v, w\rangle \qquad (\text{for all } v \in D)$$

then the adjoint of $T-\lambda$ can be defined on w simply as $(T-\lambda)^*w = 0$, since

$$\langle Tv, w\rangle \;=\; 0 \;=\; \langle v, 0\rangle \qquad (\text{for all } v \in D)$$

Thus, $T^* = T$ is defined on w, and $Tw = \overline{\lambda}w$. For λ not real, this implies $w = 0$. Thus, $(T-\lambda)D$ is *dense* in V.

Since T is self-adjoint, it is *closed*, so $T-\lambda$ is closed. The equality

$$|(T-\lambda)v|^2 \;=\; |(T-x)v|^2 + y^2|v|^2$$

gives

$$|(T-\lambda)v|^2 \;\ll_y\; |v|^2$$

Thus, for fixed $y \neq 0$, the map

$$v \oplus (T-\lambda)v \;\longrightarrow\; (T-\lambda)v$$

respects the metrics, in the sense that

$$|(T-\lambda)v|^2 \;\le\; |(T-\lambda)v|^2+|v|^2 \;\ll_y\; |(T-\lambda)v|^2 \qquad \text{(for fixed } y\neq 0)$$

The graph of $T-\lambda$ is *closed*, so is a *complete* metric subspace of $V\oplus V$. Since F respects the metrics, it preserves completeness. Thus, the metric space $(T-\lambda)D$ is *complete*, so is a closed subspace of V. Since the closed subspace $(T-\lambda)D$ is dense, it is V. Thus, for $\lambda\notin\mathbb{R}$, R_λ is everywhere-defined. Its norm is bounded by $1/|\mathrm{Im}\,\lambda|$, so it is a continuous linear operator on V.

Similarly, for T *positive*, for $\mathrm{Re}\,(\lambda)\le 0$,

$$|(T-\lambda)v|^2 \;=\; |Tv|^2-\lambda\langle Tv,v\rangle-\overline{\lambda}\langle v,Tv\rangle+|\lambda|^2\cdot|v|^2 \;=\; |Tv|^2+2|\mathrm{Re}\,\lambda|\langle Tv,v\rangle+|\lambda|^2\cdot|v|^2 \;\ge\; |\lambda|^2\cdot|v|^2$$

Then the same argument proves the existence of an everywhere-defined inverse $R_\lambda=(T-\lambda)^{-1}$, with $\|R_\lambda\|\le 1/|\lambda|$ for $\mathrm{Re}\,\lambda\le 0$. ///

[0.0.12] Theorem: (Hilbert) For points λ,μ off the real line, or, for T *positive*, for λ,μ off $[0,+\infty)$,

$$R_\lambda-R_\mu \;=\; (\lambda-\mu)R_\lambda R_\mu$$

For the operator-norm topology, $\lambda\to R_\lambda$ is *holomorphic* at such points.

Proof: Applying R_λ to

$$1_V-(T-\lambda)R_\mu \;=\; \big((T-\mu)-(T-\lambda)\big)R_\mu \;=\; (\lambda-\mu)R_\mu$$

gives

$$R_\lambda(1_V-(T-\lambda)R_\mu) \;=\; R_\lambda\big((T-\mu)-(T-\lambda)\big)R_\mu \;=\; R_\lambda(\lambda-\mu)R_\mu$$

Then

$$\frac{R_\lambda-R_\mu}{\lambda-\mu} \;=\; R_\lambda R_\mu$$

For holomorphy, with $\lambda\to\mu$,

$$\frac{R_\lambda-R_\mu}{\lambda-\mu}-R_\mu^2 \;=\; R_\lambda R_\mu-R_\mu^2 \;=\; (R_\lambda-R_\mu)R_\mu \;=\; (\lambda-\mu)R_\lambda R_\mu R_\mu$$

Taking operator norm, using $\|R_\lambda\|\le 1/|\mathrm{Im}\,\lambda|$,

$$\Big\|\frac{R_\lambda-R_\mu}{\lambda-\mu}-R_\mu^2\Big\| \;\le\; \frac{|\lambda-\mu|}{|\mathrm{Im}\,\lambda|\cdot|\mathrm{Im}\,\mu|^2}$$

Thus, for $\mu\notin\mathbb{R}$, as $\lambda\to\mu$, this operator norm goes to 0, demonstrating the holomorphy.

For *positive* T, the estimate $\|R_\lambda\|\le 1/|\lambda|$ for $\mathrm{Re}\,\lambda\le 0$ yields holomorphy on the negative real axis by the same argument. ///

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