It is amazing that resolvents \( R_\lambda = (T - \lambda)^{-1} \) exist, as everywhere-defined, continuous linear maps on a Hilbert space, even for \( T \) unbounded, and only densely-defined. Of course, some further hypotheses on \( T \) are needed, but these hypotheses are met in useful situations occurring in practice.

In particular, we will need that \( T \) is symmetric, in the sense that \( \langle Tv, w \rangle = \langle v, Tw \rangle \) for \( v, w \) in the domain of \( T \). And we will need to replace \( T \) by its Friedrichs extension, described explicitly below. For example, the Friedrichs extension replaces genuine differentiation by \( L^2 \)-differentiation.\[1\]

So-called unbounded operators on a Hilbert space \( V \) are not literally operators on \( V \), being defined on proper subspaces of \( V \). For unbounded operators on \( V \), the actual domain is an essential part of a description: an unbounded operator \( T \) on \( V \) is a subspace \( D \) of \( V \) and a linear map \( T : D \to V \). The interesting case is that the domain \( D \) is dense in \( V \).

The linear map \( T \) is most likely not continuous when \( D \) is given the subspace topology from \( V \), or it would extend by continuity to the closure of \( D \), presumably \( V \).

Explicit naming of the domain of an unbounded operator is often suppressed, instead writing \( T_1 \subset T_2 \) when \( T_2 \) is an extension of \( T_1 \), in the sense that the domain of \( T_2 \) contains that of \( T_1 \), and the restriction of \( T_2 \) to the domain of \( T_1 \) agrees with \( T_1 \).

An operator \( T', D' \) is a sub-adjoint to an operator \( T, D \) when

\[
\langle Tv, w \rangle = \langle v, T'w \rangle \quad \text{for} \quad v \in D, \ w \in D'
\]

For \( D \) dense, for given \( D' \) there is at most one \( T' \) meeting the adjointness condition.

The adjoint \( T^* \) is the unique maximal element, in terms of domain, among all sub-adjoints to \( T \). That there is a unique maximal sub-adjoint requires proof, given below.

An operator \( T \) is symmetric when \( T \subset T^* \), and self-adjoint when \( T = T^* \). These comparisons refer to the domains of these not-everywhere-defined operators. In the following claim and its proof, the domain of a map \( S \) on \( V \) is incorporated in a reference to its graph

\[
\text{graph } S = \{v \oplus Sv : v \in \text{domain } S\} \subset V \oplus V
\]

[0.0.1] Remark: In practice, anticipating that a given unbounded operator is self-adjoint when extended suitably, a simple version of the operator is defined on an easily described, small, dense domain, specifying a symmetric operator. Then a self-adjoint extension is shown to exist, as in Friedrichs’ theorem below.

[0.0.2] Remark: A symmetric operator that fails to be self-adjoint is necessarily unbounded, since bounded symmetric operators are self-adjoint, because of the existence of orthogonal complements in Hilbert spaces. The latter idea is applied to not-necessarily-bounded operators in the following.

[1] Friedrichs 1934 construction of suitable extensions predates [Sobolev 1937,1938], though the extensions use an abstracted version of what nowadays are usually called Sobolev spaces. The physical motivation for the construction is energy estimates. Existence results for self-adjoint extensions had been discussed in [Neumann 1929], [Stone 1929,30,34], but a useful description of a natural extension first occurred in [Friedrichs 1934]. Further, a Hilbert-space precursor of the Lax-Milgram theorem of [Lax-Milgram 1954] also appears in [Friedrichs 1934], following by the argument Friedrichs uses to prove that his construction gives an extension.
The direct sum \( V \oplus V \) is a Hilbert space, with natural inner product
\[
\langle v \oplus w, v' \oplus w' \rangle = \langle v, v' \rangle + \langle w, w' \rangle
\]
Define an isometry \( U \) of \( V \oplus V \) by
\[
U : V \oplus V \rightarrow V \oplus V \quad \text{by} \quad v \oplus w \rightarrow -w \oplus v
\]

[0.0.3] **Claim:** Given \( T \) with dense domain \( D \), there is a unique maximal \( T^*, D^* \) among all sub-adjoints to \( T, D \). Further, the adjoint \( T^* \) is closed, in the sense that its graph is closed in \( V \oplus V \). In fact, the adjoint is characterized by its graph, which is the orthogonal complement in \( V \oplus V \) to an image of the graph of \( T \), namely,
\[
\text{graph } T^* = \text{orthogonal complement of } U(\text{graph } T)
\]

**Proof:** The adjointness condition \( \langle Tv, w \rangle = \langle v, T^* w \rangle \) for given \( w \in V \) is an orthogonality condition
\[
\langle w \oplus T^* w, U(v \oplus Tv) \rangle = 0 \quad \text{(for all } v \text{ in the domain of } T)\]
Thus, the graph of any sub-adjoint is a subset of
\[
X = U(\text{graph } T)^\perp
\]
Since \( T \) is densely-defined, for given \( w \in V \) there is at most one possible value \( w' \) such that \( w \oplus w' \in X \), so this orthogonality condition determines a well-defined function \( T^* \) on a subset of \( V \), by
\[
T^* w = w' \quad \text{(if there exists } w' \in V \text{ such that } w \oplus w' \in X)\]
The linearity of \( T^* \) is immediate. It is maximal among sub-adjoints to \( T \) because the graph of any sub-adjoint is a subset of the graph of \( G^* \). Orthogonal complements are closed, so \( T^* \) has a closed graph. \(///\)

[0.0.4] **Corollary:** For \( T_1 \subset T_2 \) with dense domains, \( T_2^* \subset T_1^* \), and \( T_1 \subset T_1^{\ast\ast} \). \(///\)

[0.0.5] **Corollary:** A self-adjoint operator has a closed graph. \( /// \)

[0.0.6] **Remark:** The closed-ness of the graph of a self-adjoint operator is essential in proving existence of resolvents, below.

[0.0.7] **Remark:** The use of the term symmetric in this context is potentially misleading, but standard. The notation \( T = T^* \) allows an inattentive reader to forget non-trivial assumptions on the domains of the operators. The equality of domains of \( T \) and \( T^* \) is understandably essential for legitimate computations.

[0.0.8] **Proposition:** Eigenvalues for symmetric operators \( T, D \) are real.

**Proof:** Suppose \( 0 \neq v \in D \) and \( Tv = \lambda v \). Then
\[
\lambda \langle v, v \rangle = \langle \lambda v, v \rangle = \langle Tv, v \rangle = \langle v, T^* v \rangle \quad \text{(because } v \in D \subset D^*)
\]
Further, because \( T^* \) agrees with \( T \) on \( D \),
\[
\langle v, T^* v \rangle = \langle v, \lambda v \rangle = \overline{\lambda} \langle \bar{v}, v \rangle
\]
Thus, \( \lambda \) is real. \( /// \)
0.0.9 Definition: A densely-defined symmetric operator $T, D$ is positive (or non-negative) when
\[ \langle Tv, v \rangle \geq 0 \quad \text{(for all } v \in D) \]

Certainly all the eigenvalues of a positive operator are non-negative real.

0.0.10 Theorem: (Friedrichs) A positive, densely-defined, symmetric operator $T, D$ has a positive self-adjoint extension.

Proof: Define a new hermitian form $\langle \cdot, \cdot \rangle_1$ and corresponding norm $\| \cdot \|_1$ by
\[ \langle v, w \rangle_1 = \langle v, w \rangle + \langle Tv, w \rangle \quad \text{(for } v, w \in D) \]

The symmetry and non-negativity of $T$ make this positive-definite hermitian on $D$. Note that $\langle v, w \rangle_1$ makes sense whenever at least one of $v, w$ is in $D$.

Let $D_1$ be the closure in $V$ of $D$ with respect to the metric $d_1$ induced by $\| \cdot \|_1$. We claim that $D_1$ is also the $d_1$-completion of $D$. Indeed, for $v_i$ a $d$-Cauchy sequence in $D$, $v_i$ is Cauchy in $V$ in the original topology, since
\[ |v_i - v_j| \leq |v_i - v_j|_1 \]

For two sequences $v_i, w_j$ with the same $d$-limit $v$, the $d$-limit of $v_i - w_i$ is 0. Thus,
\[ |v_i - w_i| \leq |v_i - w_i|_1 \to 0 \]

For $h \in V$ and $v \in D_1$, the functional $\lambda_h : v \to \langle v, h \rangle$ has a bound
\[ |\lambda_h v| \leq |v| \cdot |h| \leq |v|_1 \cdot |h| \]

Thus, the norm of the functional $\lambda_h$ on $D_1$ is at most $|h|$. By Riesz-Fischer, there is unique $Bh$ in the Hilbert space $D_1$ with $|Bh|_1 \leq |h|$, such that
\[ \lambda_h v = \langle Bh, v \rangle_1 \quad \text{(for } v \in D_1) \]

Thus,
\[ |Bh| \leq |Bh|_1 \leq |h| \]

The map $B : V \to D_1$ is verifiably linear. There is an obvious symmetry of $B$:
\[ \langle Bv, w \rangle = \lambda_w Bv = \langle Bv, Bw \rangle_1 = \langle Bw, Bv \rangle_1 = \overline{\lambda_B w} = \overline{\langle Bw, v \rangle} = \langle v, Bw \rangle \quad \text{(for } v, w \in V) \]

Positivity of $B$ is similar:
\[ \langle Bv, v \rangle = \lambda_w Bv = \langle Bv, Bv \rangle_1 \geq \langle Bv, Bv \rangle = 0 \]

Next, $B$ has dense image in $D_1$: for $w \in D_1$ such that $\langle Bh, w \rangle_1 = 0$ for all $h \in V$, $0 = \langle w, Bh \rangle = \lambda_h w = \langle h, w \rangle \quad \text{(for all } h \in V) \]

Thus, $w = 0$, proving density of the image of $B$ in $D_1$. Finally $B$ is injective: if $Bw = 0$, then for all $v \in D_1$
\[ 0 = \langle v, 0 \rangle_1 = \langle v, Bw \rangle_1 = \lambda_w v = \langle v, w \rangle \]

Thus, $B$ is self-adjoint.

Since $B$ injects $V$ to a dense subset $D_1$, necessarily $A$ surjects from its domain (inside $D_1$) to $V$. We claim that $A$ is self-adjoint. Let $S : V \oplus V \to V \oplus V$ by $S(v \oplus w) = w \oplus v$. Then

$$\text{graph } A = S(\text{graph } B)$$

Also, in computing orthogonal complements $X^\perp$, clearly

$$(SX)^\perp = S(X^\perp)$$

From the obvious $U \circ S = -S \circ U$, compute

$$\text{graph } A^* = (U \text{ graph } A)^\perp = (U \circ S \text{ graph } B)^\perp = (-S \circ U \text{ graph } B)^\perp$$

$$= -S((U \text{ graph } B)^\perp) = -\text{ graph } A = \text{ graph } A$$

since the domain of $B^*$ is the domain of $B$. Thus, $A$ is self-adjoint.

We claim that for $v$ in the domain of $A$, $\langle Av, v \rangle \geq \langle v, v \rangle$. Indeed, letting $v = Bw$,

$$\langle v, Av \rangle = \langle Bw, w \rangle = w_1 Bw = \langle Bw, Bw \rangle_1 \geq \langle Bw, Bw \rangle = \langle v, v \rangle$$

Similarly, with $v' = Bw'$, and $v \in D_1$,

$$\langle v, Av' \rangle = \langle v, w' \rangle = \lambda_w Bw = \langle v, Bw' \rangle_1 = \langle v, v' \rangle_1 \quad (v \in D_1, v' \text{ in the domain of } A)$$

Since $B$ maps $V$ to $D_1$, the domain of $A$ is contained in $D_1$. We claim that the domain of $A$ is dense in $D_1$ in the $d$-topology, not merely in the subspace topology from $V$. Indeed, for $v \in D_1$ $\langle, \rangle_1$-orthogonal to the domain of $A$, for $v'$ in the domain of $A$, using the previous identity,

$$0 = \langle v, v' \rangle_1 = \langle v, Av' \rangle$$

Since $B$ injects $V$ to $D_1$, $A$ surjects from its domain to $V$. Thus, $v = 0$.

Last, prove that $A$ is an extension of $S = 1_V + T$. On one hand, as above,

$$\langle v, Sw \rangle = \lambda_w w = \langle v, BSw \rangle_1 \quad (\text{for } v, w \in D)$$

On the other hand, by definition of $\langle, \rangle_1$,

$$\langle v, Sw \rangle = \langle v, w \rangle_1 \quad (\text{for } v, w \in D)$$

Thus,

$$\langle v, w - BSw \rangle_1 = 0 \quad (\text{for all } v, w \in D)$$

Since $D$ is $d$-dense in $D_1$, $BSw = w$ for $w \in D$. Thus, $w \in D$ is in the range of $B$, so is in the domain of $A$, and

$$Aw = A(BSw) = Sw$$

Thus, the domain of $A$ contains that of $S$ and extends $S$.

Let $R_\lambda = (T - \lambda)^{-1}$ for $\lambda \in \mathbb{C}$ when this inverse exists as a linear operator defined at least on a dense subset of $V$. 

4
\[ [0.0.11] \textbf{Theorem:} \text{ Let } T \text{ be self-adjoint and densely defined. For } \lambda \in \mathbb{C}, \lambda \not\in \mathbb{R}, \text{ the operator } R_\lambda \text{ is everywhere defined on } V, \text{ and the operator norm is estimated by } \]

\[ |R_\lambda| \leq \frac{1}{|\text{Im } \lambda|} \]

For \( T \) positive, for \( \lambda \not\in [0, +\infty), \) \( R_\lambda \) is everywhere defined on \( V, \) and the operator norm is estimated by

\[ |R_\lambda| \leq \begin{cases} 
\frac{1}{|\text{Im } \lambda|} & \text{(for } \text{Re}(\lambda) \leq 0) \\
\frac{1}{|\lambda|} & \text{(for } \text{Re}(\lambda) \geq 0) 
\end{cases} \]

\[ \text{Proof: For } \lambda = x + iy \text{ off the real line and } v \text{ in the domain of } T, \]

\[ |(T - \lambda)v|^2 = |(T + x)v|^2 + \langle (T - x)v, iyv \rangle + \langle iyv, (T - x)v \rangle + y^2|v|^2 \]

\[ = |(T + x)v|^2 - iy((T - x)v, v) + iy\langle v, (T - x)v \rangle + y^2|v|^2 \]

The symmetry of \( T \), and the fact that the domain of \( T^* \) contains that of \( T \), implies that

\[ \langle v, Tv \rangle = \langle T^*v, v \rangle = \langle Tv, v \rangle \]

Thus,

\[ |(T - \lambda)v|^2 = |(T - x)v|^2 + y^2|v|^2 \geq y^2|v|^2 \]

Thus, for \( y \neq 0 \), \( (T - \lambda)v \neq 0 \). Let \( D \) be the domain of \( T \). On \((T - \lambda)D\) there is an inverse \( R_\lambda \) of \( T - \lambda \), and for \( w = (T - \lambda)v \) with \( v \in D \),

\[ |w| = |(T - \lambda)v| \geq |y| \cdot |v| = |y| \cdot |R_\lambda(T - \lambda)v| = |y| \cdot |R_\lambda w| \]

which gives

\[ |R_\lambda w| \leq \frac{1}{|\text{Im } \lambda|} \cdot |w| \quad (\text{for } w = (T - \lambda)v, v \in D) \]

Thus, the operator norm on \((T - \lambda)D\) satisfies \( |R_\lambda| \leq 1/|\text{Im } \lambda| \) as claimed.

We must show that \((T - \lambda)D\) is the whole Hilbert space \( V \). If

\[ 0 = \langle (T - \lambda)v, w \rangle \quad (\text{for all } v \in D) \]

then the adjoint of \( T - \lambda \) can be defined on \( w \) simply as \((T - \lambda)^*w = 0 \), since

\[ \langle Tv, w \rangle = 0 = \langle v, 0 \rangle \quad (\text{for all } v \in D) \]

Thus, \( T^* = T \) is defined on \( w \), and \( Tw = \overline{\lambda}w \). For \( \lambda \) not real, this implies \( w = 0 \). Thus, \((T - \lambda)D\) is dense in \( V \).

Since \( T \) is self-adjoint, it is \textit{closed}, so \((T - \lambda)D\) is closed. The equality

\[ |(T - \lambda)v|^2 = |(T - x)v|^2 + y^2|v|^2 \]

gives

\[ |(T - \lambda)v|^2 \ll_y |v|^2 \]

Thus, for fixed \( y \neq 0 \), the map

\[ v \oplus (T - \lambda)v \longrightarrow (T - \lambda)v \]
respects the metrics, in the sense that
\[ |(T - \lambda)v|^2 \leq |(T - \lambda)v|^2 + |v|^2 \ll_y |(T - \lambda)v|^2 \quad \text{(for fixed } y \neq 0) \]

The graph of \( T - \lambda \) is closed, so is a complete metric subspace of \( V \oplus V \). Since \( F \) respects the metrics, it preserves completeness. Thus, the metric space \((T - \lambda)D\) is complete, so is a closed subspace of \( V \). Since the closed subspace \((T - \lambda)D\) is dense, it is \( V \). Thus, for \( \lambda \notin \mathbb{R}, R_\lambda \) is everywhere-defined. Its norm is bounded by \( 1/|\text{Im} \lambda| \), so it is a continuous linear operator on \( V \).

Similarly, for \( T \) positive, for \( \text{Re} \lambda \leq 0 \),
\[ |(T - \lambda)v|^2 = |v|^2 - \lambda \langle Tv, v \rangle + |\lambda|^2 \cdot |v|^2 = |Tv|^2 + 2|\text{Re} \lambda| \langle Tv, v \rangle + |\lambda|^2 \cdot |v|^2 \geq |\lambda|^2 \cdot |v|^2 \]
Then the same argument proves the existence of an everywhere-defined inverse \( R_\lambda = (T - \lambda)^{-1} \), with \( \|R_\lambda\| \leq 1/|\lambda| \) for \( \text{Re} \lambda \leq 0 \).

[0.0.12] Theorem: (Hilbert) For points \( \lambda, \mu \) off the real line, or, for \( T \) positive, for \( \lambda, \mu \) off \( [0, +\infty) \),
\[ R_\lambda - R_\mu = (\lambda - \mu)R_\lambda R_\mu \]

For the operator-norm topology, \( \lambda \to R_\lambda \) is holomorphic at such points.

Proof: Applying \( R_\lambda \) to
\[ 1_V - (T - \lambda)R_\mu = ((T - \mu) - (T - \lambda))R_\mu = (\lambda - \mu)R_\mu \]
gives
\[ R_\lambda(1_V - (T - \lambda)R_\mu) = R_\lambda((T - \mu) - (T - \lambda))R_\mu = R_\lambda(\lambda - \mu)R_\mu \]
Then
\[ \frac{R_\lambda - R_\mu}{\lambda - \mu} = R_\lambda R_\mu \]
For holomorphy, with \( \lambda \to \mu \),
\[ \frac{R_\lambda - R_\mu}{\lambda - \mu} - R_\mu^2 = R_\lambda R_\mu - R_\mu^2 = (R_\lambda - R_\mu)R_\mu = (\lambda - \mu)R_\lambda R_\mu R_\mu \]
Taking operator norm, using \( \|R_\lambda\| \leq 1/|\text{Im} \lambda| \),
\[ \left\| \frac{R_\lambda - R_\mu}{\lambda - \mu} - R_\mu^2 \right\| \leq \frac{|\lambda - \mu|}{|\text{Im} \lambda| \cdot |\text{Im} \mu|^2} \]
Thus, for \( \mu \notin \mathbb{R}, \) as \( \lambda \to \mu \), this operator norm goes to 0, demonstrating the holomorphy.

For positive \( T \), the estimate \( \|R_\lambda\| \leq 1/|\lambda| \) for \( \text{Re} \lambda \leq 0 \) yields holomorphy on the negative real axis by the same argument. ///


