[0.0.1] **Theorem:**  
(Fujisaki’s Lemma) For a number field $k$ let

$$J^1 = \{ \alpha \in J_k : |\alpha| = 1 \}$$

Then the quotient $k^\times \setminus J^1$ is compact.

**Proof:**  
Give $\mathbb{A} = \mathbb{A}_k$ a Haar measure so that $k \setminus \mathbb{A}$ has measure 1. First, we have the Minkowski-like claim that a compact subset $C$ of $\mathbb{A}$ with measure greater than 1 cannot inject to the quotient $k \setminus \mathbb{A}$. Indeed, suppose, to the contrary, that $C$ injects to the quotient. Letting $f$ be the characteristic function of $C$,

$$1 < \int_{\mathbb{A}} f(x) \, dx = \int_{k \setminus \mathbb{A}} \sum_{\gamma \in k} f(\gamma + x) \, dx \leq \int_{k \setminus \mathbb{A}} 1 \, dx = 1$$

(last inequality by injectivity)

contradiction, proving the claim.

Fix compact $C \subset \mathbb{A}$ with measure greater than 1. For idele $\alpha$, the change-of-measure on $\mathbb{A}$ is

$$\frac{d(\alpha x)}{dx} = |\alpha|$$

Thus, neither $\alpha C$ nor $\alpha^{-1} C$ inject to the quotient $k \setminus \mathbb{A}$.

So there are $x \neq y$ in $k$ so that $x + \alpha C = y + \alpha C$. Subtracting, $x - y \in \alpha(C - C) \cap k$. Since $x - y \neq 0$ and $k$ is a field, $k^\times \cap \alpha(C - C) \neq \emptyset$. Likewise, $k^\times \cap \alpha^{-1}(C - C) \neq \emptyset$.

Thus, there are $a, b$ in $k^\times$ such that

$$a \cdot \alpha \in C - C \quad b \cdot \alpha^{-1} \in C - C$$

There is an obvious constraint

$$ab = (a \cdot \alpha)(b \cdot \alpha^{-1}) \in (C - C)^2 \cap k^\times = \text{compact} \cap \text{discrete} = \text{finite}$$

Let $\Xi$ be the latter finite set. That is, given $|\alpha| = 1$, there is $a \in k^\times$ such that $a \cdot \alpha \in C - C$, and $\xi \in \Xi$ ($\xi$ is ab just above) such that $\xi a^{-1} \cdot \alpha^{-1} \in C - C$. That is,

$$(a \cdot \alpha, (a \cdot \alpha^{-1}) \in (C - C) \times \Xi^{-1}(C - C)$$

The topology on $\mathbb{J}$ is obtained by imbedding $\mathbb{J} \to \mathbb{A} \times \mathbb{A}$ by $\alpha \to (\alpha, \alpha^{-1})$ and taking the subset topology. Thus, for each $\xi \in \Xi$,

$$\left((C - C) \times \xi^{-1}(C - C)\right) \cap \mathbb{J} = \text{compact in } \mathbb{J}$$

The continuous image in $k^\times \setminus \mathbb{J}$ of each of these finitely-many compacts is compact, and their union covers the closed subset $k^\times \setminus J^1$, so the latter is compact.

**Exercise:** Adapt the proof to treat *division algebras* $k$: one must keep track of left and right more scrupulously than was done above.