Inducing cuspidal representations from compact opens
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We follow
to see how to construct some supercuspidal representations of p-adic reductive groups from cuspidal (in a slightly different sense) representations of a maximal compact subgroup.

The further idea that all supercuspidal representations appear in this fashion occurred in
• R. Howe, Tamely ramified supercuspidal representations of \( GL(n) \), Pac. J. Math. 73 (1977), 437-460.
• R. Howe, Some qualitative results on the representation theory of \( GL(n) \) over a p-adic field, Pac. J. Math. 73 (1977), 479-538.

This was brought to a certain fruition in

Let \( G \) be a p-adic reductive group, with special maximal compact \( K \). For example, \( G = GL(n, \mathbb{Q}_p) \) and \( K = GL(n, \mathbb{Z}_p) \). Let \( \sigma \) be a an irreducible representation of \( K \) with the cuspidal property that, for every parabolic \( P \) of \( G \) and for \( N \) the unipotent radical of \( P \)

\[
\int_{N \cap K} \sigma(n) \, dn = 0 \in \text{End}_\mathbb{C}(\sigma)
\]

For \( G = GL(2, \mathbb{Q}_p) \) and \( K = GL(2, \mathbb{Z}_p) \), for example, let \( H \) be the normal subgroup of \( K \) consisting of matrices congruent to 1 mod \( p \), so we have \( GL(2, \mathbb{Z}/p) \approx K/H \). Then finite group theory (explicit counting of conjugacy classes versus the irreducibles constructed via parabolic induction, etc.) shows that that \( GL(2, \mathbb{Z}/p) \) has many cuspidal representations in this sense. That is, since there are \( (q + 1)(q - 1) \) conjugacy classes there are \( (q + 1)(q - 1) \) distinct irreducibles, of which \( (q^2 - q)/2 \) are cuspidal.\(^1\)

Let \( Z \) be the center of \( G \), and extend \( \sigma \) to \( Z \cap K \). Let

\[
\pi = \text{Ind}_{ZK}^G \sigma
\]

be the uniformly locally constant induction (the smooth dual to the compactly-supported induction). Fix a choice \( A \) of maximal split torus in a choice of minimal parabolic.

**Proposition:** For \( f \in \pi \), for sufficiently small \( \varepsilon > 0 \), for \( a \in A^{-}(\varepsilon) \) we have

\[
f(a^{-1}) = 0
\]

Thus, \( a \to f(a^{-1}) \) has compact support on \( A \) modulo the center \( Z \).

**Proof:** Let \( \Delta^+ \) be the positive roots on \( A \) corresponding to the choice of minimal parabolic. For \( \varepsilon > 0 \), let

\[
A^{-}(\varepsilon) = \{ a \in A : |\alpha(a)| < \varepsilon, \text{ for all } \alpha \in \Delta^+ \}
\]

\(^1\) The count of cuspidal representations is not completely trivial, but does follow from counting the irreducible principal series, and the special representations and one-dimensional representations that occur in the reducible principal series.
E.g., see http://www.math.umn.edu/~garrett/m/v/toy_GL2.dvi

\(^2\) Such an extension exists because \( K \cap Z \) is open in \( Z \), so \( Z/Z \cap K \) is discrete.
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Call another parabolic standard if it contains that fixed minimal one. Let \( f \in \pi^{K'} \) for a compact open subgroup \( K' \). For sufficiently small \( \varepsilon > 0 \), for all unipotent radicals \( N \) of standard parabolics,

\[
a(N \cap K')a^{-1} \supset N \cap K \quad \text{for all } a \in A^-(\varepsilon)
\]

Since \( f \) is right \( K' \)-invariant it is certainly right \((K' \cap N)\)-invariant. Then

\[
f(a^{-1}) = \int_{N \cap K} f(a^{-1}n) dn = \int_{N \cap K'} f(a^{-1}na^{-1}) dn = \int_{N \cap K'} \sigma(a^{-1}na) dn \cdot f(a^{-1})
\]

Replacing \( n \) by \( ana^{-1} \), up to a change-of-measure constant this is

\[
\int_{a^{-1}(N \cap K')a} \sigma(n) dn \cdot f(a^{-1}) = \int_{a^{-1}(N \cap K')a/(N \cap K)} \left( \int_{K \cap N} \sigma(n') dn' \right) dn' \cdot f(a^{-1})
\]

The inner integral is 0, since \( \sigma \) is cuspidal on \( K \). Thus,

\[
f(t^{-1}) = 0
\]

for \( t \in A^-(\varepsilon) \) depending on \( K' \).

**Theorem:** Every function \( f \in \pi \) is compactly supported modulo the center \( Z \) of \( G \).

**Proof:** Let \( f \in \pi^{K'} \). For fixed compact open \( K' \), let \( X \) be a (finite) collection of representatives for \( K/K' \).

By the Cartan decomposition

\[
G = KAK = \bigcup_x KAxK'
\]

Let

\[
A^+ = \{ a \in A : |\alpha(a)| \geq 1, \text{ for all } \alpha \in \Delta^+ \}
\]

\[
A^- = \{ a \in A : |\alpha(a)| \leq 1, \text{ for all } \alpha \in \Delta^+ \}
\]

Since \( K \) contains (representatives for) the Weyl group \( W \) of \( A \), we have

\[
A = \bigcup_{w \in W} wA^+w^{-1}
\]

so in fact

\[
G = KA^+K = \bigcup_x KAxK'
\]

For \( x \in X \), the function \( f_x(g) = f(gx) \) is in \( \pi^{xK'x^{-1}} \). Thus, for \( x \in X \) there is \( \varepsilon_x \) such that \( f(a^{-1}x = 0 \) for \( a \in A^-\varepsilon_x \). Let \( \varepsilon > 0 \) be the minimum of all the \( \varepsilon_x \). Then for any \( x \in X \), \( k \in K \), \( k' \in K' \), for all \( a \in A^-\varepsilon \),

\[
f(ka^{-1}xk') = 0
\]

The set

\[
C_{K'} = A^- - A^-\varepsilon
\]

is compact modulo the center, and \( f \) is 0 off \( KC_{K'}K \), which is compact modulo the center since \( K \) is compact.

**Corollary:** \( \pi \) is admissible: for a compact open subgroup \( K' \) of \( G \),

\[
\dim_{\mathbb{C}} \pi^{K'} < \infty
\]
Proof: Again, any set
\[ C_{K'} = A^- - A^-(\varepsilon) \]
is compact modulo \( Z \), and \( f \) is zero off some set \( K C_{K'} K \), and by the compactness mod \( Z \)
\[ KC_{K'} K / Z K' = \text{finite} \]
A function \( f \in \pi_{K'} \) is well-defined on such a quotient, so lies in a finite-dimensional space. ///

Theorem: The induced representation \( \pi \) is supercuspidal.

Proof: We show that all Jacquet modules (co-isotypes of the trivial representation of \( N \))
\[ \pi_N = \pi / (\text{subspace generated by all } v - \pi(n) \cdot v, n \in N, v \in \pi) \]
are 0, for \( N \) the unipotent radical of a standard parabolic \( P = MN \) (a Levi decomposition). Given \( f \in \pi \),
by the Iwasawa decomposition, there are compacta \( C_M \subset M \) and \( C_N \subset N \) such that
\[ \text{spt} f \subset K Z C_M C_N \]
Given \( m \in Z C_M \), take \( N' \) a compact open subgroup of \( N \) such that \( N' \supset C_N \) and
\[ N' \supset \bigcup_{m \in Z C_M} m^{-1}(N \cap K)m \]
This is possible since the latter union is compact, being a continuous image of the compact \( C_M \times N \), noting that \( Z \) acts trivially by conjugation. For \( g \in G \) let \( g = kmn \) with \( k \in K, m \in Z C_M \), and \( n \in C_N \). Then
\[ \int_{N'} f(gn') \, dn' = \int_{N'} f(kmmn') \, dn' = \sigma(k) \cdot \int_{N'} f(mnn') \, dn' \]
And since \( N' \supset N_C \), we can replace \( n' \) by \( n^{-1} n' \), and the integral becomes
\[ \int_{N'} f(mn') \, dn' = \int_{N'} f(mn'm^{-1} \cdot m) \, dn' \]
Since \( mN'm^{-1} \supset N \cap K \), this is
\[ \int_{(N \cap K)N'} \left( \int_{N \cap K} f(n \cdot mn'm^{-1} \cdot m) \, dn \right) \, dn' \]
The inner integral is
\[ \int_{N \cap K} \sigma(n) \, dn \cdot f(mn'm^{-1} \cdot m) = 0 \cdot f(mn'm^{-1} \cdot m) \]
by the cuspidality of \( \sigma \). Thus, the whole is 0. ///